出 版 前 言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰 100 周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of Circular Cylinder (II)

PART (III)

Effect of an Elliptic Hole

Ref Coker & Filon Photoelasticity pp 540-544

If we write

$$\chi_1 = e^{2\xi} + \cos 2\eta$$

$$\chi_2 = e^{-2\xi} + \cos 2\eta$$

$$\chi_3 = e^{-2\xi} \cos 2\eta$$

$$\chi_4 = \xi$$

$$\chi_5 = e^{2\xi} \cos 2\eta$$

then it is found that the stress function can be written as

$$\chi = \frac{1}{16} T \left\{ \chi_1 + (2e^{2\alpha} - 1)\chi_2 - e^{4\alpha}\chi_3 + 4(1 - \cosh 2\alpha)\chi_4 - \chi_5 \right\}$$

Now the stresses given by the different stress functions are, if we write $(\cosh 2\xi - \cos 2\eta) = 2J^2$,

$$\begin{cases} 2J^4 \widehat{\xi\xi}_1 = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{4\xi} \\ 2J^4 \widehat{\eta\eta}_1 = \cos 4\eta - 4\cos 2\eta\, e^{2\xi} + 2 + e^{4\xi} \\ 2J^4 \widehat{\xi\eta}_1 = 2\sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_2} = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\eta\eta_2} = \cos 4\eta - 4\cos 2\eta\; e^{-2\xi} + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\xi\eta_2} = -2\sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_3} = \cos 4\eta \cdot e^{-2\xi} - \cos 2\eta\,(e^{-4\xi}+3) + 3e^{-2\xi} \\ 2J^4\,\widehat{\eta\eta_3} = -\cos 4\eta \cdot e^{-2\xi} - 3e^{-2\xi} + \cos 2\eta\,(e^{-4\xi}+3) \\ 2J^4\,\widehat{\xi\eta_3} = \sin 4\eta\, e^{-2\xi} - \sin 2\eta\,(e^{-4\xi}+3) \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_4} = \sinh 2\xi \\ 2J^4\,\widehat{\eta\eta_4} = -\sinh 2\xi \\ 2J^4\,\widehat{\xi\eta_4} = \sin 2\eta \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_5} = \cos 4\eta\, e^{2\xi} - \cos 2\eta\,(e^{4\xi}+3) + 3e^{2\xi} \\ 2J^4\,\widehat{\eta\eta_5} = -\cos 4\eta\, e^{2\xi} + \cos 2\eta\,(e^{4\xi}+3) - 3e^{2\xi} \\ 2J^4\,\widehat{\eta\xi_5} = -\sin 4\eta\, e^{2\xi} + \sin 2\eta\,(e^{4\xi}+3) \end{cases}$$

To find the strain energy increase in the specimen, it is best to find the increase in work done by the external forces, because the difficulty of carrying out the integration in elliptical coordinates.

We have

$$\left\{\begin{array}{l} 2\mu J u_1 = (2-4\sigma)e^{2\xi} - (4-4\sigma)\cos 2\eta \\ 2\mu J v_1 = -(2-4\sigma)\sin 2\eta \end{array}\right\} \text{ due to } X_1$$

$$\left\{\begin{array}{l} 2\mu J u_2 = (4-4\sigma)\cos 2\eta - (2-4\sigma)e^{-2\xi} \\ 2\mu J v_2 = -(2-4\sigma)\sin 2\eta \end{array}\right\} \text{ due to } X_2$$

$$\left\{\begin{array}{l} 2\mu J u_3 = 2e^{-2\xi}\cos 2\eta \\ 2\mu J v_3 = 2e^{-2\xi}\sin 2\eta \end{array}\right\} \text{ due to } X_3$$

$$\{\ 2\mu J v_4 = -1 \qquad \text{ due to } X_4$$

$$\left\{\begin{array}{l} 2\mu J v_5 = -2e^{2\xi}\cos 2\eta \\ 2\mu J v_5 = 2e^{2\xi}\sin 2\eta \end{array}\right\} \text{ due to } X_5$$

Therefore the total displacements

$$2\mu J u = \frac{1}{16} T \Big[(2-4\sigma) e^{2\xi} - (4-4\sigma)\cos 2\eta + (2e^{2\alpha}-1)\{(4-4\sigma)\cos 2\eta - (2-4\sigma)e^{-2\xi}\}$$
$$- e^{4\alpha}\, 2 e^{-2\xi}\cos 2\eta - 4(1-\cosh 2\alpha) + 2 e^{2\xi}\cos 2\eta \Big]$$

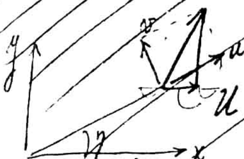
$$2\mu J v = \frac{1}{16} T \Big[-(2-4\sigma)\sin 2\eta - (2e^{2\alpha}-1)(2-4\sigma)\sin 2\eta - 2e^{4\alpha} e^{-2\xi}\sin 2\eta$$
$$- 2 e^{2\xi}\sin 2\eta \Big]$$

or

$$2\mu J u = \frac{T}{8} \Big[(1-2\sigma) e^{2\xi} - (2-2\sigma)\cos 2\eta + (2e^{2\alpha}-1)\{(2-2\sigma)\cos 2\eta - (1-2\sigma)e^{-2\xi}\}$$
$$- e^{4\alpha} e^{-2\xi}\cos 2\eta - 2(1-\cosh 2\alpha) + e^{2\xi}\cos 2\eta \Big]$$

$$2\mu J v = -\frac{T}{8}\sin 2\eta \Big[(1-2\sigma) + (2e^{2\alpha}-1)(1-2\sigma) + e^{4\alpha} e^{-2\xi} + e^{2\xi} \Big]$$

The component displacements in the direction of tension:



$$U = u\cos\eta - v\sin\eta$$

the uniform tension

$$\chi_0 = \frac{1}{2} T y^2 = \frac{T}{2} \sinh^2\xi \sin^2\eta$$

$$= \frac{T}{8} (\cosh 2\xi - 1)(1 - \cos 2\eta)$$

$$= \frac{T}{16} \left\{ (e^{2\xi} + \cos 2\eta) + (e^{-2\xi} + \cos 2\eta) - e^{2\xi}\cos 2\eta - e^{-2\xi}\cos 2\eta - 2 \right\}$$

$$= \frac{T}{16} \left\{ \chi_1 + \chi_2 - \chi_3 - \chi_5 \right\}$$

Thus the displacements

$$2\mu J u_0 = \frac{T}{8} \left\{ 2(1-2\sigma) \sinh 2\xi + 2 \sinh 2\xi \cos 2\eta \right\}$$

$$2\mu J v_0 = -\frac{T}{8} \left\{ 2(1-2\sigma) \sin 2\eta + 2 \cosh 2\xi \sin 2\eta \right\}$$

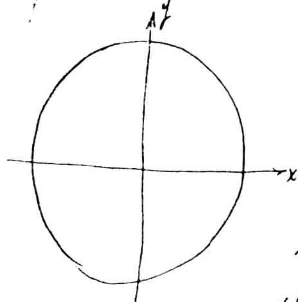
$$= -\frac{T}{8} \sin 2\eta \left\{ 2(1-2\sigma) + 2\cosh 2\xi \right\}$$

~~$\widehat{\xi\xi} = T\cos\eta \quad \widehat{\xi\eta} = -T\sin\eta$ when $\xi = \infty$~~

$$2\mu J(u - u_0) = \frac{T}{8} \left\{ (1-2\sigma) e^{-2\xi} - 2(1-\sigma)\cos 2\eta + (2e^{2\alpha} - 1)\left\{ 2(1-\sigma)\cos 2\eta - (1-2\sigma) e^{-2\xi} \right\} - e^{4\alpha} e^{-2\xi} \cos 2\eta - 2(1 - \cosh 2\alpha) + e^{-2\xi} \cos 2\eta \right\}$$

$$2\mu J(v - v_0) = -\frac{T}{8} \sin 2\eta \left\{ 2(e^{2\alpha} - 1) + e^{4\alpha} e^{-2\xi} - e^{-2\xi} \right\}$$

Now consider the circle at infinity



$$\sqrt{x^2+y^2} = \frac{c}{2} e^{\xi}$$

$$J^2 = \frac{1}{2}\frac{1}{2} e^{2\xi} c^2, \quad \therefore \quad J = \frac{c}{2} e^{\xi}$$

The work done by external forces will be the shear $\widehat{\xi\eta}$ + $\widehat{\xi\xi}$.

$$\widehat{\xi\eta} = -\frac{1}{2} T \sin 2\eta$$

$$\widehat{\xi\xi} = \frac{1}{2} T (1 + \cos 2\eta)$$

Increase in strain energy

$$= \frac{T}{32\mu}\frac{T}{2}\left[\int_0^{2\pi} (1+\cos 2\eta)\left\{4(1-\sigma)(e^{2\alpha}-1)\cos 2\eta - 2(1-\cosh 2\alpha)\right\} d\eta \right.$$

$$\left. + \int_0^{2\pi} \sin^2 2\eta \cdot 2(e^{2\alpha}-1)\, d\eta \right]$$

$$= \frac{T^2}{64\mu}\pi \left[4(1-\sigma)(e^{2\alpha}-1) + 4(\cosh 2\alpha - 1) + 2(e^{2\alpha}-1)\right]$$

$$= \frac{T^2}{32\mu}\pi \left[(3-2\sigma)(e^{2\alpha}-1) + 2(\cosh 2\alpha - 1)\right]$$

$$= \frac{T^2}{16\mu}\pi \left[(3-2\sigma) e^{\alpha} \sinh \alpha + (\cosh 2\alpha - 1)\right] c^2$$

Increase in strain energy $\mathcal{E}$

$$= \frac{(1+\sigma)T^2}{16E}\pi c^2 \left[(3-2\sigma)(\sinh\alpha + \cosh\alpha)\sinh\alpha + (\cosh 2\alpha - 1) \right]$$

The axis of the ellipse,

$$a = c\cosh\alpha \qquad a^2 - b^2 = c^2$$
$$b = c\sinh\alpha$$

$$= t\frac{(1+\sigma)T^2}{16E}\pi \left[(3-2\sigma)(b+a)b + 2b^2 \right]$$

$$\mathcal{E} = \frac{(1+\sigma)T^2\pi t}{16E}\left[(5-2\sigma)b^2 + (3-2\sigma)ab \right]$$

$$= \frac{(1+\sigma)T^2}{16E}(\pi ab)t\left[(3-2\sigma) + (5-2\sigma)\left(\frac{b}{a}\right) \right] \qquad \text{O.K.}$$

It is thus shown that the presence of a hole always increases the total strain energy, even compared with the whole flat plate. Therefore we have too much restraining a buckling stress can only be arrived by considering more accurately the interaction.

Now for the sake of simplicity, go back to the case of a circular buckled region. Here, in order that the buckled circular plate be clamp supported, we choose the form of buckling to be

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}^2$$

Thus

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r \partial \theta} = \frac{1}{R}\frac{\partial^2 w_0}{\partial r \partial \theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{R}\frac{\partial^2 w_0}{\partial \theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}\left(\frac{r}{R}\right)^2 \frac{1}{r} \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2\right\}\frac{1}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

$$\frac{1}{r^4}\left(\frac{\partial w}{\partial\theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w_0}{\partial\theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r\,\partial\theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w_0}{\partial r\,\partial\theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}\left(\sin^2\theta\right)^2 - \frac{1}{R^2}\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\right]\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{1}{R^2}\left[8f\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta - 16f^2\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right\}$$

$$= \frac{1}{R^2}\cos 2\theta \cdot 4f\left\{\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right\}$$

$$\nabla^4\varphi = \frac{4Ef}{R^2}\left[2\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta + \cos 2\theta\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right) - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{4Ef}{R^2}\left[\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right) - \left(\frac{r^2}{R^2}\right)\cos 2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

If we put $\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\sin^2\theta}{R^2} - f\left(\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2}{2}\right)^2$

$$\nabla^4\varphi = C\left[p^2-4\right]\left[(p-2)^2-4\right] r^{p-4}\cos 2\theta$$

$p = 6,$
$$\nabla^4\varphi = C\cdot 32\cdot 12\, r^2\cos 2\theta$$

$\therefore$
$$C = \frac{-K}{384}$$

$\therefore$
$$\varphi_4 = -\frac{Ef}{384R^2}\frac{r^6}{R^2}\cos 2\theta$$

Therefore the particular integral is

$$\boxed{\varphi_0 = \frac{EfR^2}{64}\left[\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\frac{r^4}{R^4} + \frac{2}{9}\left(2f\frac{a^2}{R^2}-1\right)\frac{r^6}{R^6} - \frac{1}{6}\frac{r^6}{R^6}\cos 2\theta - \frac{1}{12}f\frac{r^8}{R^8}\right]}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\nabla^4\varphi = 0$$

can be written as

$$\frac{\varphi}{R^2} = \left(\frac{Ef}{64}\right)\left[\frac{1}{4}B_0\frac{r^2}{R^2} + S_0 + \cos 2\theta\left[P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 \cdots\right] + \cancel{\cos 4\theta\left[P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right]} + \cos 6\theta\left[P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right]\right]$$

$$\frac{\varphi}{R^2} = \left(\frac{Ef}{64}\right)\Bigg[\left\{ + \frac{1}{4}Q_0\left(\frac{r}{R}\right)^2 + \left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^4 + \frac{2}{9}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^6 - \frac{1}{12}f\left(\frac{r}{R}\right)^8\right\}$$

$$+ \cos 2\theta \left\{ P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 + \quad - \frac{1}{6}\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 4\theta \left\{ P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 6\theta \left\{ P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right\}$$

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} = \left(\frac{Ef}{64}\right)\Bigg[\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\right\}$$

$$+ \cos 2\theta \left\{ 2P_2 + 4R_2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4\right\}$$

$$+ \cos 4\theta \left\{ 4P_4\left(\frac{r}{R}\right)^2 + 6R_4\left(\frac{r}{R}\right)^4\right\}$$

$$+ \cos 6\theta \left\{ 6P_6\left(\frac{r}{R}\right)^4 + 8R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{Ef}{64}\Bigg[-4\cos 2\theta\left\{P_2 + R_2\left(\frac{r}{R}\right)^2 - \frac{1}{6}\left(\frac{r}{R}\right)^4\right\}$$

$$- 16\cos 4\theta\left\{P_4\left(\frac{r}{R}\right)^2 + R_4\left(\frac{r}{R}\right)^4\right\}$$

$$- 36\cos 6\theta\left\{P_6\left(\frac{r}{R}\right)^4 + R_6\left(\frac{r}{R}\right)^6\right\}\Bigg]$$

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\Big\}$$

$$- \cos 2\theta\Big\{\ 2P_2 + \frac{1}{3}\left(\frac{r}{R}\right)^4\Big\}$$

$$- \cos 4\theta\Big\{\ 12 P_4\left(\frac{r}{R}\right)^2 + 10 R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$- \cos 6\theta\Big\{\ 30 P_6\left(\frac{r}{R}\right)^4 + 28 R_6\left(\frac{r}{R}\right)^6\Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}f\left(\frac{r}{R}\right)^6\Big\}$$

$$+ \cos 2\theta\Big\{P_2 + 12 R_2\left(\frac{r}{R}\right)^2 - 5\left(\frac{r}{R}\right)^4\Big\} \qquad !!!$$

$$+ \cos 4\theta\Big\{12 P_4\left(\frac{r}{R}\right)^2 + 30 P_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+ \cos 6\theta\Big\{30 P_6\left(\frac{r}{R}\right)^4 + 56 R_6\left(\frac{r}{R}\right)^6\Big\}$$

$$\widehat{r\theta} = \left(\frac{Ef}{64}\right)\Big[\ 2\sin 2\theta\Big\{P_2 + 3 R_2\left(\frac{r}{R}\right)^2 - \frac{5}{6}\left(\frac{r}{R}\right)^4\Big\}$$

$$+ 4\sin 3\theta\Big\{3 P_4\left(\frac{r}{R}\right)^2 + 5 R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+ 6\sin 6\theta\Big\{5 P_6\left(\frac{r}{R}\right)^4 + 7 R_6\left(\frac{r}{R}\right)^6\Big\}\Big]$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left(\widehat{rr} - \nu\,\widehat{\theta\theta}\right)$$

$$\therefore \quad \frac{u}{R} = \frac{p}{64}\Bigg[\Big\{(1-\nu)\frac{1}{2}Q_0\left(\frac{r}{R}\right) + \frac{4}{3}(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^3$$

$$+\frac{4}{15}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^5 - \frac{2}{21}(1-7\nu)f\left(\frac{r}{R}\right)^7\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2\left(\frac{r}{R}\right) + 4\nu R_2\left(\frac{r}{R}\right)^3 + \frac{1}{15}(1-15\nu)\left(\frac{r}{R}\right)^5\Big\}$$

$$-\cos 4\theta\Big\{4(1+\nu)P_4\left(\frac{r}{R}\right)^3 + 2(1+3\nu)R_4\left(\frac{r}{R}\right)^5\Big\}$$

$$-\cos 6\theta\Big\{6(1+\nu)P_6\left(\frac{r}{R}\right)^5 + 4(1+2\nu)R_6\left(\frac{r}{R}\right)^7\Big\} + F(\theta)$$

$$\frac{1}{E}\left(\widehat{\theta\theta} - \nu\,\widehat{rr}\right) = \frac{p}{64}\Bigg[\Big\{(1-\nu)\frac{Q_0}{2} + 4(3-\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+\frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\Big\{(1+2\nu)P_2 + 12R_2\left(\frac{r}{R}\right)^2 - \left(5-\frac{1}{3}\nu\right)\left(\frac{r}{R}\right)^4\Big\}$$

$$+\cos 4\theta\Big\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2 + 10(3+\nu)R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+\cos 6\theta\Big\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4 + 28(2+\nu)R_6\left(\frac{r}{R}\right)^6\Big\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}+\frac{\partial u}{\partial r}=\frac{1}{E}\left(\sigma_r-\nu\sigma_\theta\right)$$

here

$$\frac{1}{E}\left(\sigma_r-\nu\sigma_\theta\right)=\frac{f^2}{64}\Bigg[\Big\{(1-\nu)\frac{Q_0}{2}+4(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+\frac{4}{3}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4-\frac{2}{3}(1-7\nu)f^2\left(\frac{r}{R}\right)^6\Big\}$$

$$-\cos 2\theta\left\{(2+\nu)P_2+12\nu R_2\left(\frac{r}{R}\right)^2-\left(\frac{1}{3}-5\nu\right)\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 4\theta\left\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2+10(1+3\nu)R_4\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 6\theta\left\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4+98(1+2\nu)R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}=\frac{1}{2}\left[\left\{\left(\frac{r}{R}\right)\sin^2\theta-4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r}{R}\right\}^2-\left\{\frac{r}{R}\sin^2\theta\right\}^2\right]$$

$$=\frac{1}{2}\left(\frac{r}{R}\right)^2 4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\left[4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)-2\sin^2\theta\right] \quad * \; 4f=f$$

$$=\frac{f}{2}\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r^2}{R^2}\left[f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)+\cos 2\theta-1\right]$$

$$=\frac{f}{64}\Bigg[\left\{32\frac{a^2}{R^2}\left(f\frac{a^2}{R^2}-1\right)\frac{r^2}{R^2}-32\left(2f\frac{a^2}{R^2}-1\right)\frac{r^4}{R^4}+32f\left(\frac{r}{R}\right)^6\right\}$$

$$+\cos 2\theta\left\{32\frac{a^2}{R^2}\frac{r^2}{R^2}-32\frac{r^4}{R^4}\right\}\Bigg]$$

$$\frac{\partial u}{\partial r} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2$$

$$+ \frac{20}{3}(5-\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^6\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2 + 4\Big(8\frac{a^2}{R^2} + 3\nu P_2\Big)\Big(\frac{r}{R}\Big)^2 - \Big(\frac{97}{3} - 5\nu\Big)\Big(\frac{r}{R}\Big)^4\Big\}$$

$$-\cos 4\theta\Big\{12(1+\nu)P_4\Big(\frac{r}{R}\Big)^2 + 10(1+5\nu)R_4\Big(\frac{r}{R}\Big)^4\Big\}$$

$$-\cos 6\theta\Big\{30(1+\nu)P_6\Big(\frac{r}{R}\Big)^4 + 28(1+9\nu)R_6\Big(\frac{r}{R}\Big)^6\Big\}$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2}\Big(\frac{r}{R}\Big) + 4(3-\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^3 + \frac{4}{3}(5-\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^5$$

$$- \frac{2}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^7\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2\Big(\frac{r}{R}\Big) + \frac{4}{3}\Big(8\frac{a^2}{R^2} + 3\nu P_2\Big)\Big(\frac{r}{R}\Big)^3 - \Big(\frac{97}{15} - \nu\Big)\Big(\frac{r}{R}\Big)^5\Big\}$$

$$-\cos 4\theta\Big\{4(1+\nu)P_4\Big(\frac{r}{R}\Big)^3 + 2(1+3\nu)R_4\Big(\frac{r}{R}\Big)^5\Big\}$$

$$-\cos 6\theta\Big\{6(1+\nu)P_6\Big(\frac{r}{R}\Big)^5 + 4(1+9\nu)R_6\Big(\frac{r}{R}\Big)^7\Big\}\Big] + F(\theta)$$

$\therefore$

$$\frac{v}{R} = \frac{p}{64}\Big[\sin 2\theta \Big\{ \tfrac{3}{2}(1+\nu) P_2 \left(\tfrac{r}{R}\right) + 2\left(\tfrac{8}{3}\tfrac{r^2}{R^2} + \overline{3+\nu}\, Q_2\right)\left(\tfrac{r}{R}\right)^3 - \left(\tfrac{86}{15} - \tfrac{2}{3}\nu\right)\left(\tfrac{r}{R}\right)^5 \Big\}$$

$$+ \sin 4\theta \Big\{ 4(1+\nu) P_4 \left(\tfrac{r}{R}\right)^3 + 2(4+2\nu) R_4 \left(\tfrac{r}{R}\right)^5 \Big\}$$

$$+ \sin 6\theta \Big\{ 6(1+\nu) P_6 \left(\tfrac{r}{R}\right)^5 + 2(5+3\nu) R_6 \left(\tfrac{r}{R}\right)^7 \Big\}\Big] - \int F(\theta)\,d\theta + G\left(\tfrac{r}{R}\right)$$

$$\frac{\int F(\theta)d\theta}{\left(\frac{r}{R}\right)} + \frac{F'(\theta)}{\left(\frac{r}{R}\right)} + G'\left(\tfrac{r}{R}\right) - \frac{G\left(\frac{r}{R}\right)}{\left(\frac{r}{R}\right)} = 0$$

or $$\int F(\theta)d\theta + F'(\theta) = G\left(\tfrac{r}{R}\right) - \tfrac{r}{R} G'\left(\tfrac{r}{R}\right)$$

or $$\int F(\theta)d\theta + F'(\theta) = C$$

$$G\left(\tfrac{r}{R}\right) - \tfrac{r}{R} G'\left(\tfrac{r}{R}\right) = C$$

$\therefore F = G = 0$

or $$F(\theta) + F''(\theta) = 0$$

$$F'' + F = 0 \quad \text{only} \quad F = A \begin{matrix}\sin\theta\\ \cos\theta\end{matrix} > \text{cut}$$

$\frac{1}{3} \times \frac{2}{3}\nu$

$$G'\left(\tfrac{r}{R}\right) - \frac{1}{\left(\frac{r}{R}\right)} G\left(\tfrac{r}{R}\right) = -\frac{C}{\left(\frac{r}{R}\right)}$$

$$\left(\tfrac{r}{R}\right)\frac{d}{d\left(\frac{r}{R}\right)}\left[\frac{1}{\left(\frac{r}{R}\right)} G\left(\tfrac{r}{R}\right)\right] = -\frac{C}{\left(\frac{r}{R}\right)} \quad \Big\| \quad \frac{1}{\left(\frac{r}{R}\right)} G = \frac{C}{\left(\frac{r}{R}\right)} + B$$

$$G = C + B\left(\tfrac{r}{R}\right)$$

The undisturbed stress function outside the circular region

$$\frac{\phi_1}{R^2} = \frac{1}{4}\sigma\left(1-\cos 2\theta\right)\left(\frac{r}{R}\right)^2$$

The other possible solutions are

$$\frac{\phi_2}{R^2} = \sigma\Bigg[P_0 \log\left(\frac{r}{R}\right) + \cos 2\theta\left\{ Q_2 \frac{1}{\left(\frac{r}{R}\right)^2} + S_2 \right\}$$

$$+ \cos 4\theta \left\{ Q_4 \frac{1}{\left(\frac{r}{R}\right)^4} + S_4 \frac{1}{\left(\frac{r}{R}\right)^2} \right\}$$

$$+ \cos 6\theta \left\{ Q_6 \frac{1}{\left(\frac{r}{R}\right)^6} + S_6 \frac{1}{\left(\frac{r}{R}\right)^4} \right\} \Bigg]$$

$$\frac{1}{\left(\frac{r}{R}\right)} \frac{\partial\left(\frac{\phi_2}{R^2}\right)}{\partial\left(\frac{r}{R}\right)} = \sigma\Bigg[P_0 \frac{1}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ - \frac{2Q_2}{\left(\frac{r}{R}\right)^4} \right\}$$

$$+ \cos 4\theta \left\{ - \frac{4Q_4}{\left(\frac{r}{R}\right)^6} - \frac{2S_4}{\left(\frac{r}{R}\right)^4} \right\}$$

$$+ \cos 6\theta \left\{ - \frac{6Q_6}{\left(\frac{r}{R}\right)^8} - \frac{4S_6}{\left(\frac{r}{R}\right)^6} \right\} \Bigg]$$

$$\frac{1}{\left(\frac{r}{R}\right)^2} \frac{\partial^2\left(\frac{\phi_2}{R^2}\right)}{\partial \theta^2} = \sigma\Bigg[-4\cos 2\theta \left\{ \frac{Q_2}{\left(\frac{r}{R}\right)^4} + \frac{S_2}{\left(\frac{r}{R}\right)^2} \right\} - 16\cos 4\theta \left\{ \frac{Q_4}{\left(\frac{r}{R}\right)^6} + \frac{S_4}{\left(\frac{r}{R}\right)^4} \right\}$$

$$- 36\cos 6\theta \left\{ \frac{Q_6}{\left(\frac{r}{R}\right)^8} + \frac{S_6}{\left(\frac{r}{R}\right)^6} \right\} \Bigg]$$

$$rr_1 = \sigma\left[\frac{R_0}{(\frac{r}{R})^2} - \cos 2\theta\left\{\frac{6Q_2}{(\frac{r}{R})^4} + \frac{4S_2}{(\frac{r}{R})^2}\right\}\right.$$

$$\left. - \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{18S_4}{(\frac{r}{R})^4}\right\} - \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{40S_6}{(\frac{r}{R})^6}\right\}\right]$$

$$\theta\theta_1 = \sigma\left[-\frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta \cdot \frac{6Q_2}{(\frac{r}{R})^4} + \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{6S_4}{(\frac{r}{R})^4}\right\}\right.$$

$$\left. + \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{20S_6}{(\frac{r}{R})^6}\right\}\right]$$

$$r\theta_2 = \sigma\left[-2\sin 2\theta\left\{\frac{3Q_2}{(\frac{r}{R})^4} + \frac{S_2}{(\frac{r}{R})^2}\right\} - 4\sin 4\theta\left\{\frac{5Q_4}{(\frac{r}{R})^6} + \frac{3S_4}{(\frac{r}{R})^4}\right\}\right.$$

$$\left. - 6\sin 6\theta\left\{\frac{7Q_6}{(\frac{r}{R})^8} + \frac{5S_6}{(\frac{r}{R})^6}\right\}\right]$$

Therefore $$rr = \sigma\left[\frac{1}{2} + \frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta\left\{\frac{1}{2} - \frac{6Q_2}{(\frac{r}{R})^4} - \frac{4S_2}{(\frac{r}{R})^2}\right\}\right.$$

$$\left. - \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{18S_4}{(\frac{r}{R})^4}\right\} - \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{40S_6}{(\frac{r}{R})^6}\right\}\right]$$

$$\theta\theta = \sigma\left[\frac{1}{2} - \frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta\left\{\frac{6Q_2}{(\frac{r}{R})^4} - \frac{1}{2}\right\} + \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{6S_4}{(\frac{r}{R})^4}\right\}\right.$$

$$\left. + \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{20S_6}{(\frac{r}{R})^6}\right\}\right]$$

$$\widehat{r\theta} = -\sigma\left[\sin 2\theta\left\{\frac{1}{2} + \frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{2S_2}{\left(\frac{a}{R}\right)^2}\right\} + \sin 4\theta\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{12S_4}{\left(\frac{a}{R}\right)^4}\right\}\right.$$ 306

$$\left. + \sin 6\theta\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{30S_6}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

The stress conditions at the boundary of the circular region are then

$$\sigma\left\{\frac{1}{2} + \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (1)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{4S_2}{\left(\frac{a}{R}\right)^2} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{2P_2 + \frac{1}{3}\left(\frac{a}{R}\right)^4\right\} \quad (2)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{18S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 10R_4\left(\frac{a}{R}\right)^4\right\} \quad (3)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{40S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 28R_6\left(\frac{a}{R}\right)^6\right\} \quad (4)$$

$$\sigma\left\{\frac{1}{2} - \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 12\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^2 - \frac{17}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (5)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{P_2 + 12R_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4\right\} \quad (6)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{6S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 30R_4\left(\frac{a}{R}\right)^4\right\} \quad (7)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{20S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 56R_6\left(\frac{a}{R}\right)^6\right\} \quad (8)$$

$$-\sigma\left\{\frac{1}{2}+\frac{6Q_2}{\left(\frac{a}{r}\right)^4}+\frac{2S_2}{\left(\frac{a}{r}\right)^2}\right\}=\frac{Ef}{64}\left\{2P_2+6R_2\left(\frac{a}{r}\right)^2-\frac{5}{3}\left(\frac{a}{r}\right)^4\right\} \tag{9}$$

$$-\sigma\left\{\frac{30\,Q_4}{\left(\frac{a}{r}\right)^6}+\frac{12\,S_4}{\left(\frac{a}{r}\right)^4}\right\}=\frac{Ef}{64}\left\{12\,P_4\left(\frac{a}{r}\right)^2+30\,R_4\left(\frac{a}{r}\right)^{[illegible]}\right\} \tag{10}$$

$$-\sigma\left\{\frac{42\,Q_6}{\left(\frac{a}{r}\right)^8}+\frac{30\,S_6}{\left(\frac{a}{r}\right)^6}\right\}=\frac{Ef}{64}\left\{30\,P_6\left(\frac{a}{r}\right)^4+42\,R_6\left(\frac{a}{R}\right)^6\right\} \tag{11}$$

$$\frac{u}{R}=\frac{\sigma}{E}\Big[\frac{1}{2}(1-\nu)\left(\frac{a}{R}\right)-(1+\nu)P_0\frac{1}{\left(\frac{a}{R}\right)}+\cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)+2Q_2(1+\nu)\frac{1}{\left(\frac{a}{R}\right)^3}+\frac{4S_2}{\left(\frac{a}{R}\right)}\right\}$$

$$+\cos 4\theta\left\{4(1+\nu)\frac{Q_4}{\left(\frac{a}{R}\right)^5}+2(3+\nu)\frac{S_4}{\left(\frac{a}{R}\right)^3}\right\}+\cos 6\theta\left\{6(1+\nu)\frac{Q_6}{\left(\frac{a}{R}\right)^7}+4(2+\nu)\frac{S_6}{\left(\frac{a}{R}\right)^5}\right\}\Big]$$

$$\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)=\frac{\sigma}{E}\Big[\frac{1}{2}(1-\nu)-(1+\nu)P_0\frac{1}{\left(\frac{a}{R}\right)^2}+\cos 2\theta\left\{-\frac{1}{2}(1+\nu)+\frac{6(1+\nu)Q_2}{\left(\frac{a}{R}\right)^4}+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}$$

$$+\cos 4\theta\left\{\frac{30(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{6(1+3\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\}+\cos 6\theta\left\{\frac{42(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{30(1+2\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}\Big]$$

$$\frac{v}{R}=\frac{\sigma}{E}\Big[\sin 2\theta\left\{\frac{2(1+\nu)Q_2}{\left(\frac{a}{R}\right)^3}\qquad-\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)\right\}$$

$$+\sin 4\theta\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^5}+\frac{4\nu\,S_4}{\left(\frac{a}{R}\right)^3}\right\}+\sin 6\theta\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^7}+\frac{(2+6\nu)S_6}{\left(\frac{a}{R}\right)^5}\right\}\Big]$$

The boundary conditions at the periphery of the circular region is then

$$* \quad \sigma\left\{\frac{1}{2}(1-\nu) - (1+\nu)\frac{P_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{(1-\nu)\frac{Q_0}{2} + 4(3-\nu)\left(\frac{a}{R}\right)^4\left(1-f\frac{a^2}{R^2}\right) + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^6\right\} \qquad (12)$$

(5 − ν [illegible])

$$-\sigma\left\{\frac{1+\nu}{2} + \frac{2Q_2(1+\nu)}{\left(\frac{a}{R}\right)^4} + \frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{(2+\nu)P_2 + \frac{4}{3}\left(f\frac{a^2}{R^2}+3\nu R_2\right)\left(\frac{a}{R}\right)^2 - \left(\frac{97}{15}-\nu\right)\left(\frac{a}{R}\right)^4\right\} \qquad (13)$$

$$-\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6} + \frac{2(3+\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2 + 2(1+3\nu)R_4\left(\frac{a}{R}\right)^4\right\} \qquad (14)$$

$$-\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8} + \frac{4(2+\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4 + 4(1+2\nu)R_6\left(\frac{a}{R}\right)^6\right\} \qquad (15)$$

$$\sigma\left\{\frac{2(1+\nu)Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}(1+\nu)\right\} = \frac{Ef}{64}\left\{\frac{3}{2}(1+\nu)P_2 + 2\left(\frac{f}{3}\frac{a^2}{R^2} + \overline{3+\nu}\,R_2\right)\left(\frac{a}{R}\right)^2 - \left(\frac{86}{15}-\frac{2}{3}\nu\right)\left(\frac{a}{R}\right)^4\right\} \qquad (16)$$

$$\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6} + \frac{4\nu S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2 + 2(4+2\nu)R_4\left(\frac{a}{R}\right)^4\right\} \qquad (17)$$

$$\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8} + \frac{2(1+3\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4 + 2(5+3\nu)R_6\left(\frac{a}{R}\right)^6\right\} \qquad (18)$$

Let us investigate the equations (2), (6), (9), (13) & (16)

$$6q_2 + 4s_2 - \frac{1}{2} = \left\{ 3p_2' + \frac{7}{3}h \right\}$$

$$h = \left(\frac{a}{R}\right)^2$$

$$6q_2 - \frac{1}{2} = -\left\{ p_2' + 12r_2' - 5h' \right\}$$

$$-\left(6q_2 + 2s_2 + \frac{1}{2}\right) = \left\{ +2p_2' + 6r_2' - \frac{5}{3}h' \right\}$$

$$-\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right] = \left\{ (2+\nu)p_2' + 4\nu r_2' + \left(\frac{21}{5} + \nu\right)h \right\}$$

$$\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu)\right] = \left\{ \frac{3}{2}(1+\nu)p_2' + 2(3+\nu)r_2' - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h \right\}$$

The question is whether this system of equations are consistent. They are not consistent, so we can only satisfy them approximately, by means of method of least square, thus

$$6\left(6q_2 + 4s_2 - \frac{1}{2} - 3p_2 - \frac{1}{3}h\right) + 6\left(6q_2 - \frac{1}{2} - p_2 - 12r_2 + 5h\right)$$

$$+ 6\left(6q_2 + 2s_2 + \frac{1}{2} + 2p_2 + 6r_2 - \frac{5}{3}h\right)$$

$$+ 2(1+\nu)\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2 + (2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5} + \nu\right)h\right]$$

$$+ 2(1+\nu)\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu) - \frac{3}{2}(1+\nu)p_2 - 2(3+\nu)r_2 + \left(\frac{2}{5} - \frac{2}{3}\nu\right)h\right] = 0.$$

or

$$\left[108 + 4(1+\nu)^2\right]q_2 + \left[36 + 8(1+\nu)\right]s_2 + \left[(1-\nu^2) - 6\right]p_2 + \left[4(1+\nu)(-3+\nu) - 36\right]r_2 + \left[-3 + 18h + 8(1+\nu)\left(\frac{23}{5} + \frac{\nu}{3}\right)h\right] = 0 \qquad (A)$$

$$4\left(6q_2+4s_2-\frac{1}{2}-2p_2-\frac{h}{3}\right)+2\left(6q_2+2s_2+\frac{1}{2}+2p_2+6r_2-\frac{5}{3}h\right)$$

$$+4\left[\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2+(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right]=0$$

$$\left[36+8(1+\nu)\right]q_2+36s_2+4(1+\nu)p_2+4(3+4\nu)r_2+\left[1+2\nu+\left(\frac{162}{15}+4\nu\right)h\right]=0 \quad (B)$$

$$2\left(2p_2+\frac{h}{3}-6q_2-4s_2+\frac{1}{2}\right)+\left(p_2+12r_2-5h-6q_2+\frac{1}{2}\right)$$

$$+2\left(2p_2+6r_2-\frac{5}{3}h+6q_2+2s_2+\frac{1}{2}\right)+(2+\nu)\left[(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right.$$

$$\left.+\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2\right]+\frac{3}{2}(1+\nu)\left[\frac{3}{2}(1+\nu)p_2+2(3+\nu)r_2-\left(\frac{2}{5}-\frac{2}{3}\nu\right)h\right.$$

$$\left.-2(1+\nu)q_2+\frac{1}{2}(1+\nu)\right]=0.$$

$$\left[-6+(1-\nu^2)\right]q_2+4(1+\nu)s_2+\left[9+(2+\nu)^2+\frac{9}{4}(1+\nu)^2\right]p_2+\left[24+4(2+\nu)\nu+3(1+\nu)(3+\nu)\right]r_2$$

$$+\left[\frac{5}{2}+\frac{1}{2}(1+\nu)\left(\frac{7}{2}+\frac{5}{2}\nu\right)+h\left\{\frac{2}{3}-5-\frac{10}{3}+(2+\nu)\left(\frac{21}{5}+\nu\right)-\frac{3}{2}(1+\nu)\left(\frac{2}{5}-\frac{2}{3}\nu\right)\right\}\right]=0 \quad (C)$$

$$6\left(p_2 + 12 r_2 - 5h - 6q_2 + \frac{1}{2}\right) + 3\left(2p_2 + 6r_2 - \frac{5}{3}h + 6q_2 + 2s_2 + \frac{1}{2}\right)$$

$$+ 2\nu\left[(2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5}+\nu\right)h + \frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right]$$

$$+ (3+\nu)\left[\frac{3}{2}(1+\nu)p_2 + 2(3+\nu)r_2 - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h - 2(1+\nu)q_2 + \frac{1}{2}(1+\nu)\right] = 0.$$

$$\left[-18 + 4\nu(1+\nu) - 2(1+\nu)(3+\nu)\right]q_2 + \left[6 + 8\nu\right]s_2$$

$$+ \left[12 + 2\nu(2+\nu) + \frac{3}{2}(1+\nu)(3+\nu)\right]p_2 + \left[90 + 8\nu^2 + 2(3+\nu)^2\right]r_2$$

$$+ \left[\frac{9}{2} + \nu(1+\nu) + \frac{1}{2}(1+\nu)(3+\nu) + h\left\{-35 + 2\nu\left(\frac{21}{5}+\nu\right) - 2(3+\nu)\left(\frac{1}{5} - \frac{1}{3}\nu\right)\right\}\right] = 0 \qquad (D)$$

The equations (A), (B), (C), (D) determines $\boxed{p_2, q_2, r_2, s_2}$

$$\widehat{rr}+\widehat{\theta\theta}=\sigma\left[1+\cos 2\theta\left\{-\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}\right]$$

$$\int_0^{2\pi} d\theta\left\{\left(\widehat{rr}+\widehat{\theta\theta}\right)^2-2(1+\nu)\left(\widehat{rr}\cdot\widehat{\theta\theta}-\widehat{r\theta}^2\right)\right\}$$

$$=\pi\sigma^2\left[2+\frac{16S_2^2}{\left(\frac{a}{R}\right)^4}-2(1+\nu)\left\{\frac{1}{2}-\frac{2R_0^2}{\left(\frac{a}{R}\right)^4}-\left(\frac{1}{2}-\frac{6Q_2}{\left(\frac{a}{R}\right)^4}\right)^2+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\left(\frac{1}{2}-\frac{6Q_2}{\left(\frac{a}{R}\right)^4}\right)\right.\right.$$

$$\left.\left.-\left(\frac{1}{2}+\frac{6Q_2}{\left(\frac{a}{R}\right)^4}+\frac{2S_2}{\left(\frac{a}{R}\right)^2}\right)^2\right\}\right]$$

$$=\pi\sigma^2\left[2+\frac{16S_2^2}{\left(\frac{a}{R}\right)^4}-2(1+\nu)\left\{\left(-2R_0^2-4S_2^2\right)\frac{1}{\left(\frac{a}{R}\right)^4}+\left(-24S_2Q_2-24S_2Q_2\right)\frac{1}{\left(\frac{a}{R}\right)^6}\right.\right.$$

$$\left.\left.+\left(-72Q_2^2\right)\frac{1}{\left(\frac{a}{R}\right)^8}\right\}\right]$$

E_1 - Strain energy outside the circular region − the same at uniform stress

$$=\frac{t\sigma^2}{2E}\pi\int_a^\infty r\,dr\left[\frac{16S_2^2}{\left(\frac{r}{R}\right)^4}+2(1+\nu)\left\{\frac{2(R_0^2+2S_2^2)}{\left(\frac{r}{R}\right)^4}+\frac{48S_2Q_2}{\left(\frac{r}{R}\right)^6}+\frac{72Q_2^2}{\left(\frac{r}{R}\right)^8}\right\}\right]$$

$$=\frac{t\sigma^2}{2E}\pi R^2\left[8\frac{S_2^2}{\left(\frac{a}{R}\right)^2}+2(1+\nu)\left\{\frac{(R_0^2+2S_2^2)}{\left(\frac{a}{R}\right)^2}+12\frac{S_2Q_2}{\left(\frac{a}{R}\right)^4}+12\frac{Q_2^2}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

For the extensional strain energy in the circular region

$$\widehat{rr}+\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{r}{R}\Big)^6\Big\}$$

$$+\cos 2\theta\Big\{-P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - \frac{16}{3}\Big(\frac{r}{R}\Big)^4\Big\}$$

$$\int_0^{2\pi} d\theta\Big\{(\widehat{rr}+\widehat{\theta\theta})^2 - 2(1+\nu)(\widehat{rr}\,\widehat{\theta\theta} - \widehat{r\theta}^2)\Big\}$$

$$= \pi\frac{E^2f^2}{64^2}\Big[2\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{r}{R}\Big)^6\Big\}^2$$

$$+\Big\{-P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - \frac{16}{3}\Big(\frac{r}{R}\Big)^4\Big\}^2$$

$$-2(1+\nu)\Big\{2\Big[\frac{1}{2}Q_0 + 4\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{r^2}{R^2} + \frac{4}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{2}{3}f\Big(\frac{r}{R}\Big)^6\Big]$$

$$\times\Big[\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{r^2}{R^2} + \frac{20}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}f\Big(\frac{r}{R}\Big)^6\Big]$$

$$-\Big[2P_2 + \frac{1}{3}\Big(\frac{r}{R}\Big)^4\Big]\Big[P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - 5\Big(\frac{r}{R}\Big)^4\Big] - 4\Big[P_2 + 3R_2\Big(\frac{r}{R}\Big)^2 - \frac{5}{6}\Big(\frac{r}{R}\Big)^4\Big]^2\Big\}\Big]$$

Extensional strain energy in the circular region, E_2

$$= \frac{\pi R^2 t E f^2}{2\times 64^2}\Big[2\Big\{\frac{1}{2}Q_0^2\left(\frac{a}{R}\right)^2 + 8Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + \frac{16^2}{6}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{8}{3}Q_0\left(\frac{a}{R}\right)^6\left(2f\frac{a^2}{R^2}-1\right) - \frac{4}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$+ \frac{64}{10}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} +$$

$$+ 32\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)\left(2f\frac{a^2}{R^2}-1\right) - \frac{64}{9}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12} + \frac{8\times 16}{21}f^2\left(\frac{a}{R}\right)^{14} - \frac{32\times 1.6}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12}\Big\}$$

$$+ \Big\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 - 6R_2P_2\left(\frac{a}{R}\right)^4 + \frac{16}{9}P_2\left(\frac{a}{R}\right)^6 + 24R_2^2\left(\frac{a}{R}\right)^6 - 16R_2\left(\frac{a}{R}\right)^8 + \frac{16\times 1.6}{3}\left(\frac{a}{R}\right)^{10}\Big\}$$

$$-4(1+\nu)\Big\{\frac{1}{8}Q_0^2\left(\frac{a}{R}\right)^2 + 2Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + 8\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{2}{3}Q_0\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^6 - \frac{1}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$+ \frac{16}{3}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)\left(2f\frac{a^2}{R^2}-1\right) - \frac{8}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12} + \frac{8}{9}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} - \frac{8}{3}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12}$$

$$+ \frac{2}{9}f^2\left(\frac{a}{R}\right)^{14}\Big\}$$

$$+2(1+\nu)\Big\{3P_2^2\left(\frac{a}{R}\right)^2 + 12P_2R_2\left(\frac{a}{R}\right)^4 - \frac{41}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{3}{4}R_2\left(\frac{a}{R}\right)^8 + \frac{7}{48}\left(\frac{a}{R}\right)^{10} + 9R_2^2\left(\frac{a}{R}\right)^6\Big\}$$

$$+8(1+\nu)\Big\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 + \frac{3}{2}P_2R_2\left(\frac{a}{R}\right)^4 + \frac{9}{4}R_2^2\left(\frac{a}{R}\right)^6 - \frac{5}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{5}{16}R_2\left(\frac{a}{R}\right)^8 + \frac{25}{360}\left(\frac{a}{R}\right)^{10}\Big\}\Big]$$

3/4

For the circular region

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\, 4f \left\{ \left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2 \right\}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$= \frac{1}{R}\, 4f \left\{ \left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \right\}$$

$$\tau = 0$$

$$k_1 + k_2 = \frac{1}{R}\, 8f \left\{ \left(\frac{a}{R}\right)^2 - 2\left(\frac{r}{R}\right)^2 \right\}$$

$$k_1 k_2 = \frac{1}{R^2}\, 16 f^2 \left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4 \right\}$$

The bending strain energy in the circular region

$$16 f^2 \frac{1}{24} \frac{t^3}{(1-\nu^2)} E \cdot 2\pi \int_0^{\left(\frac{a}{R}\right)} \left[4\left\{ \left(\frac{a}{R}\right)^2 - 2\left(\frac{r}{R}\right)^2 \right\}^2 - 2(1+\nu)\left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4 \right\} \right] \frac{r}{R}\, d\frac{r}{R}$$

$$= \frac{4}{3} \frac{t^3 E \pi}{(1-\nu^2)} f^2 \int_0^{\left(\frac{a}{R}\right)} \left[2\left(\frac{a}{R}\right)^4 - 8\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 8\left(\frac{r}{R}\right)^4 - (1+\nu)\left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4 \right\} \right] \frac{r}{R}\, d\frac{r}{R}$$

$$= \frac{4}{3} \frac{t^3 E \pi f^2}{(1-\nu^2)} \left[1 - 2 + \frac{4}{3} - (1+\nu)\left(\frac{1}{2} - 1 + \frac{1}{2}\right) \right] \left(\frac{a}{R}\right)^6$$

$$= \frac{8}{9} \frac{t^3 E \pi f^2}{(1-\nu^2)} \left(\frac{a}{R}\right)^6 = \mathcal{E}_3 \quad \times 16 \quad [\text{due to } \underline{4f = f}]\ !!!$$

the decrease in potential of σ [now - case of uniform stress]

$$= tR^2 \frac{\sigma^2}{E} \int_0^{2\pi} d\theta \Big[\tfrac{1}{2}(1-\nu)R_0 - \tfrac{1}{2}(1+\nu)R_0 + \cos^2\theta\big(-2(1+\nu)S_2 + 2S_2\big)\Big]$$

$$- d\theta\big[-(1+\nu)S_2\big]\sin^2\theta \Big]$$

$$= t\frac{\sigma^2}{E}\pi R^2\big[-2\nu R_0 - 2\nu S_2 + (1+\nu)S_2\big]$$

$$= t\frac{\sigma^2}{E}\pi R^2\big[(1-\nu)S_2 - 2\nu R_0\big] = f(\sigma)$$

In order to simplify the calculation: [Note: diff. from p. 309] !!!

Put $\dfrac{R_0}{\left(\frac{a}{R}\right)^2} = r_0, \qquad f\dfrac{a^2}{R^2} = g, \qquad \dfrac{Q_0}{\left(\frac{a}{R}\right)^4} = q_0$

$\dfrac{Q_2}{\left(\frac{a}{R}\right)^4} = q_2, \qquad \dfrac{S_2}{\left(\frac{a}{R}\right)^2} = s_2, \qquad \dfrac{P_2}{\left(\frac{a}{R}\right)^4} = p_2,$

$\dfrac{R_2}{\left(\frac{a}{R}\right)^2} = r_2$

Important !!! Change eqn. (A), (B), (C), (D)

With this set of notation, !!!

$$\frac{\mathcal{E}_1}{R^3} = \frac{t}{R}\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\gamma s_2^2 + 2(1+\nu)\left\{(r_0^2 + 2s_2^2) + 12\,[illegible]\,r_2 - 12\,p_2^2\right\}\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3\frac{8}{9}E\pi g^2\left(\frac{a}{R}\right)^2\frac{1}{(1-\nu^2)}\quad\frac{1}{16}$$

!!! (22-2)

$$\frac{\mathcal{W}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left\{(1-\nu)s_2 - 2\nu r_0\right\}$$

$\frac{r}{r}$!!!

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{1}{8192}E g^2\left(\frac{a}{R}\right)^6\Big[\Big\{q_0^2 + 16q_0(1-g) + \frac{256}{3}(1-g)^2 + \frac{16}{3}q_0(2g-1) - \frac{8}{3}q_0 g$$

$$+ 64(1-g)(2g-1) - \frac{128}{9}g(2g-1) + \frac{64}{3}(2g-1)^2$$

$$+ \frac{256}{21}g^2 - \frac{102.4}{3}g(1-g)\Big\}$$

$$+ \left\{\frac{1}{2}p_2^2 - 6r_2p_2 + \frac{16}{9}p_2 + 24r_2^2 - 16r_2 + \frac{25.6}{3}\right\}$$

$$-(1+\nu)\Big\{\frac{1}{2}q_0^2 + 8q_0(1-g) + 32(1-g)^2 + \frac{8}{3}q_0(2g-1) - \frac{4}{3}q_0 g$$

$$+ \frac{64}{3}(1-g)(2g-1) - \frac{32}{3}g(1-g) + \frac{32}{9}(2g-1)^2 - \frac{32}{3}g(2g-1) + \frac{8}{7}g^2\Big\}$$

$$+(1+\nu)\left\{6p_2^2 + 24p_2r_2 - \frac{49}{9}p_2 - \frac{3}{2}r_2 + \frac{7}{24} + 18r_2^2\right\}\Big]$$

The equations for determining the values of r_0 + g_0

$$\frac{1}{2} + r_0 = \left[\frac{E}{640\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} g_0 + 4(1-g) + \frac{4}{3}(2g-1) - \frac{2}{3} g \right\}$$

$$\frac{1}{2} - r_0 = \left[\frac{E}{640\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} g_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{14}{3} g \right\}$$

$$1 = \left[\frac{E}{640\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ g_0 + 16(1-g) + 8(2g-1) - \frac{16}{3} g \right\}$$

$$g_0 = \frac{1}{g} \frac{640\sigma}{E\left(\frac{a}{R}\right)^2} - 16(1-g) - 8(2g-1) + \frac{16}{3} g$$

$$\boxed{g_0 = \frac{1}{g} \frac{640\sigma}{E\left(\frac{a}{R}\right)^2} - 8 + \frac{16}{3} g}$$

If $\frac{E}{640\sigma}\left(\frac{a}{R}\right)^2 = \xi$

$$\boxed{g_0 = \frac{1}{g\xi} - 8 + \frac{16}{3} g}$$

$$r_0 = \left[\frac{E}{640\sigma} \frac{a^2}{R^2}\right] g \left\{ \frac{1}{2g} \frac{640\sigma}{E\left(\frac{a}{R}\right)^2} - 4 + \frac{8}{3} g + \frac{8}{3} - 2g \right\} - \frac{1}{2}$$

$$\boxed{r_0 = \left[\frac{E}{640\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{2}{3} g - \frac{4}{3} \right\}}$$

$$\boxed{r_0 = \xi g \left(\frac{2}{3} g - \frac{4}{3} \right)}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right) \frac{E\left(\frac{a}{R}\right)^2}{8192} \Big[\left\{ q_0^2 + \frac{4}{3}(4-3g)\, q_0 + 34.1333 - 49.7778\, g + 26.4127\, g^2 \right\}$$

$$+ \left\{ \frac{1}{3} p_2^2 - 6 r_2 p_2 + \frac{16}{9} p_2 + 27 r_2^2 - 15 r_2 + \frac{256}{3} \right\}$$

$$- (1+\nu) \left\{ \frac{1}{2} q_0^2 + 4\left(\frac{4}{3} - g\right) q_0 + 14.2222 - 14.2222\, g - 6.2222\, g^2 \right\}$$

$$+ (1+\nu) \left\{ 6 p_2^2 + 24 p_2 r_2 - \frac{49}{9} p_2 - \frac{3}{2} r_2 + 18 r_2^2 + \frac{7}{24} \right\} \Big]$$

The four equations for the unknowns r_2, s_2, q_2, p_2 are now:

$\nu = 0.3$ $\quad 1+\nu = 1.3$ $\quad (1+\nu)^2 = 1.69$

$1-\nu^2 = 0.91$

$$114.76\, q_2 + 46.4\, s_2 - 5.07(\xi g p_2) - 30.64(\xi g r_2) + [30.22\, \xi g - 3] = 0$$

$$46.4\, q_2 + 36\, s_2 + 5.20(\xi g p_2) + 16.8(\xi g r_2) + [13.3333\, \xi g + 1.6] = 0$$

$$-5.09\, q_2 + 5.2\, s_2 + 19.5625(\xi g p_2) + 39.63(\xi g r_2) + [2.29333\, \xi g + 5.2625] = 0$$

$$-25.02\, q_2 + 8.4\, s_2 + 19.815(\xi g p_2) + 122.50(\xi g r_2) + [-32.96\, \xi g + 7.035] = 0$$

or

1) $q_2 + 0.40432\, s_2 - 0.044353(\xi g p_2) - 0.43604(\xi g r_2) + [0.26333\, \xi g - 0.026142] = 0$

2) $q_2 + 0.77586\, s_2 + 0.11207(\xi g p_2) + 0.36207(\xi g r_2) + [0.28736\, \xi g + 0.034483] = 0$

2) $-q_2 + 1.02161\, s_2 + 3.84332(\xi g p_2) + 7.78585(\xi g r_2) + [0.45056\, \xi g + 1.03389] = 0$

1) $-q_2 + 0.33573\, s_2 + 0.79197(\xi g p_2) + 4.89608(\xi g r_2) + [-1.31735\, \xi g + 0.28118] = 0$

$$1.42593\, S_2 + 3.79897(\xi g\, p_2) + 7.34981(\xi g\, r_2) + [0.71389\, \xi g + 1.00775] = 0$$

$$1.79747\, S_2 + 3.95539(\xi g\, p_2) + 8.14792(\xi g\, r_2) + [0.73792\, \xi g + 1.06837] = 0$$

$$1.11159\, S_2 + 0.90404(\xi g\, p_2) + 5.25815(\xi g\, r_2) + [-1.02999\, \xi g + 0.31566] = 0$$

$$S_2 + 2.66421(\xi g\, p_2) + 5.15440(\xi g\, r_2) + [0.50065\, \xi g + 0.70673] = 0$$

$$S_2 + 2.20053(\xi g\, p_2) + 4.53299(\xi g\, r_2) + [0.41053\, \xi g + 0.59437] = 0$$

$$S_2 + 0.81329(\xi g\, p_2) + 4.73030(\xi g\, r_2) + [-0.92659\, \xi g + 0.28397] = 0$$

$$1.85092(\xi g\, p_2) + 0.42410(\xi g\, r_2) + [1.42724\, \xi g + 0.42276] = 0$$

$$0.46368(\xi g\, p_2) + 0.62141(\xi g\, r_2) + [0.09012\, \xi g + 0.11236] = 0$$

$$\xi g\, p_2 + 0.22913(\xi g\, r_2) + [0.77110\, \xi g + 0.22841] = 0$$

$$\xi g\, p_2 + 1.34017(\xi g\, r_2) + [0.19436\, \xi g + 0.24232] = 0$$

Thus $$\xi g\, r_2 = \frac{0.57674\, \xi g - 0.01391}{1.11104}$$

$$\boxed{\xi g\, r_2 = 0.51910\, \xi g - 0.01252}$$

$$(\xi g\, p_2) + 1.56930(0.51910\, \xi g - 0.01252) + (0.96546\, \xi g + 0.47073) = 0$$

$$\boxed{\xi g\, p_2 = -(0.87004\, \xi g + 0.22554)}$$

$$3S_2 - 5.67803(0.89004\,\xi_g + 0.22554) + 14.41769(0.51910\,\xi_g - 0.01252)$$
$$+ [-0.01541\,\xi_g + 1.58507] = 0$$

$$3S_2 - (5.05367\,\xi_g + 1.28062) + (7.48422\,\xi_g - 0.18051)$$
$$+ (-0.01541\,\xi_g + 1.58507) = 0$$

$$\boxed{S_2 = -(0.80505\,\xi_g + 0.04131)}$$

$$2q_2 - 1.18018(0.80505\,\xi_g + 0.04131) - 0.06772(0.89004\,\xi_g + 0.22554)$$
$$- 0.07397(0.51910\,\xi_g - 0.01252) + (0.55069\,\xi_g + 0.00834) = 0$$

$$2q_2 - (0.95010\,\xi_g + 0.04875) - (0.06027\,\xi_g + 0.01527)$$
$$- (0.03840\,\xi_g - 0.00926) + (0.55069\,\xi_g + 0.00834) = 0$$

$$\boxed{q_2 = 0.24904\,\xi_g + 0.02738}$$

We have thus

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(0.80505\,\xi g + 0.04131)^2 + 2(1+\nu)\Big\{(\xi g)^2(g-2)^2\frac{4}{9} + 2(0.80505\,\xi g + 0.04131)^2$$

$$-12(0.80505\,\xi g + 0.04131)(0.24904\,\xi g + 0.02738) + 12(0.24904\,\xi g + 0.02738)^2\Big\}\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[6.2(0.80505\,\xi g + 0.04131)^2 + 1.15556(\xi g)^2(g^2-4g+4)$$

$$- 31.20(0.24904\,\xi g + 0.02738)(0.55691\,\xi g + 0.01393)\Big]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2(1.15556\,g^2 - 4.62222\,g + 4.31326) - 0.17159(\xi g) - 0.001320\Big]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a^2}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2(34.5016\,g^2 - 31.2890\,g + 24.5569)$$

$$+(3.73333 - 2.8g)(\xi g)\Big\{1 - 8\left(1-\tfrac{2}{3}g\right)(\xi g)\Big\} + 5.3(\xi g)(0.89004\,\xi g + 0.22554)$$

$$- 17.95(\xi g)(0.5190\,\xi g - 0.01252) + 8.3(0.89004\,\xi g + 0.22554)^2$$

$$- 25.20(0.89004\,\xi g + 0.22554)(0.5190\,\xi g - 0.01252) + 47.4(0.5190\,\xi g - 0.01252)^2\Big]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\frac{a^2}{R^2}\frac{\sigma^2}{2E}\Big[(\xi g)^2(19.56813\,g^2 + 11.0221\,g - 2.2066) + \xi g(5.20071 - 2.8g) + 0.50079\Big]}$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[\frac{4096}{9(1-\nu^2)} \quad \frac{1}{\left(\frac{a}{R}\right)^4}(\xi g)^2\right]$$

323

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[\frac{1}{9(1-\nu^2)} \quad \frac{g^2}{\left(\frac{\sigma}{E}\right)^2}\right]$$

$$\boxed{\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[0.122100 \quad \frac{g^2}{\left(\frac{\sigma}{E\frac{t}{R}}\right)^2}\right]}$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[-1.4\,(0.80505\,\xi g + 0.04131) - 1.2\,\xi g\left(\frac{2}{3}g - \frac{4}{3}\right)\right]$$

$$\boxed{-\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\left[(\xi g)(0.8g - 0.47293) + 0.057834\right]}$$

Total potential of the system:

$$\left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2(20.7239\,g^2 + 6.3999\,g + 2.1067) + \xi g\,(5.0291 - 2.8\,g) + 0.49947$$
$$+ 0.122100\,\frac{g^2}{k^2} + \xi g\,(0.8g - 0.47293) + 0.057834\Big]$$

If σ is a compression, write

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\Big[\xi^2\left(20.7239\,g^4+6.3999\,g^3+2.1067\,g^2\right)+$$

$$-\ \xi\left(4.5562\,g-2.0\,g^2\right)+0.122100\,\frac{g^2}{k^2}+0.55730\Big]$$

Differentiate against g,

$$\xi^2\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)-\xi\left(4.5562-4.0\,g\right)$$

$$+\ 0.244200\,\frac{g}{k^2}=0$$

or $$K^2=\frac{0.244200\,g}{\xi(4.5562-4.0g)-\xi^2(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

or $$(4.5562-4g)=2\xi\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)$$

$$\therefore\quad \xi=\frac{1}{2}\,\frac{(4.5562-4g)}{(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

$$K^2=4\,\frac{0.244200\,g^2(82.8956\,g^2+19.1999\,g+4.2134)}{(4.5662-4g)^2}$$

Now if we minus the energy expression with the quantity $\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{9E} \cdot 1$, so that the expression truly represents the difference of total potential of the system in two modes, then

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{9E}\left[\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{20.7239\, f^4\left(\frac{a}{R}\right)^8 + 6.3999 f^3\left(\frac{a}{R}\right)^6 + 2.1067 f^2\left(\frac{a}{R}\right)^4\right\}\right.$$

$$\left. - \left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 f\left(\frac{a}{R}\right)^2 - 2.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.122100 f^2\left(\frac{a}{R}\right)^4 \frac{1}{K^2} - 0.44270\right]$$

the minimum condition becomes

$$\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{82.8956\, f^3\left(\frac{a}{R}\right)^6 + 19.1999 f^2\left(\frac{a}{R}\right)^6 + 4.2134 f\left(\frac{a}{R}\right)^4\right\}$$

$$- \left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562\left(\frac{a}{R}\right)^2 - 4f\left(\frac{a}{R}\right)^4\right\} + 0.244200 f\left(\frac{a}{R}\right)^2 \frac{1}{K^2} = 0$$

or if $\left(\frac{a}{R}\right) \neq 0$

$$\boxed{\begin{aligned}&\left(\frac{E}{64\sigma}\right)^2\left\{82.8956\, f^3\left(\frac{a}{R}\right)^6 + 19.1999 f^2\left(\frac{a}{R}\right)^4 + 4.2134\, f\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{R}\right)^4 \\ &- \left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 - 4f\left(\frac{a}{R}\right)^2\right\} + f\left(\frac{a}{R}\right)^2 \frac{0.244200}{K^2} = 0\end{aligned}}$$

$$\left(\frac{E}{640}\right)^2\left(\frac{a}{R}\right)^4\left\{145.0673 f^4\left(\frac{a}{R}\right)^8 + 38.3974 f^3\left(\frac{a}{R}\right)^6 + 10.5335 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{640}\right)\left(\frac{a}{R}\right)^2\left\{13.6686 f\left(\frac{a}{R}\right)^2 - 8.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.366300 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270 = 0$$

Due to very nature of the conditions, it is easier to proceed as follows.

$$\left(\frac{a}{R}\right)^4\left\{82.8956 g^3 + 19.1999 g^2 + 4.2134 g\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{4.5562 - 4g\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.244200\, g}{K^2} = 0$$

or

$$\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{4.5562 - 4g\right\}}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\} g}\left(\frac{a}{R}\right)^2 + \frac{0.244200 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\}} = 0$$

$$\left(\frac{a}{R}\right)^4\left\{145.0673 g^4 + 38.3974 g^3 + 10.5335 g^2\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{13.6686 g - 8g^2\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.366300 g^2}{K^2} - 0.44270 = 0$$

$$\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{13.6686 - 8g\right\}\left(\frac{a}{R}\right)^2}{\left\{145.0673 g^2 + 38.3974 g + 10.5335\right\} g} + \frac{0.36630 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3974 g + 10.5335\right\}}$$

$$- \frac{0.44270 \times 64^2 \times K^2\left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3974 g + 10.5335\right\} g^2} = 0$$

Both equations can be written as

$$\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{\underbrace{\left\{0.32381 g^2 + 0.075000 g + 0.016459\right\}}_{X} g}\left(\frac{a}{R}\right)^2 + \frac{3.9072\left(\frac{t}{R}\right)^2}{\left\{0.32381 g^2 + 0.075000 g + 0.016459\right\}} = 0$$

$$\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.70858 - g\right\}}{\underbrace{\left\{0.28333 g^2 + 0.075000 g + 0.020573\right\}}_{W} g}\left(\frac{a}{R}\right)^2 + \frac{\left(2.9304 g^2 - 3.5416 K^2\right)\left(\frac{t}{R}\right)^2}{\left\{0.28333 g^2 + 0.075000 g + 0.020573\right\} g^2} = 0$$

$$\left(\frac{a}{R}\right)^2 = \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{X g} \pm \sqrt{\frac{1}{4}\,\frac{K^2\left(\frac{t}{R}\right)^2\left\{1.13905 - g\right\}^2}{X^2 g^2} - \frac{3.9072\left(\frac{t}{R}\right)^2}{X}}$$

$$= \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{X g}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2\left\{1.13905 - g\right\}^2}}\right]$$

$$\frac{K^2\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{(1.70858 - g)}{W} - \frac{(1.13905 - g)}{X}\right]\frac{1}{2}\,\frac{(1.13905 - g)}{X}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2 (1.13905 - g)^2}}\right]$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{3.9072 g^2}{X} - \frac{\left(2.9304 g^2 - 3.5416 K^2\right)}{W}\right] = 0$$

$$\frac{1}{2}\,\frac{(1.13905-g)}{X}\left[\frac{(1.70858-g)}{W}-\frac{(1.13905-g)}{X}\right]\left[1\pm\sqrt{1-\frac{15.6288\,X g^2}{K^2(1.13905-g)^2}}\right]K^2$$

$$+\left[\frac{3.9072g^2}{X}-\frac{(2.9304g^2-3.5416K^2)}{W}\right]=0$$

where $X = 0.32381\,g^2 + 0.075000\,g + 0.016459$

$W = 0.28333\,g^2 + 0.075000\,g + 0.020573$

When $g = 0.1$

$X = 0.027197, \quad W = 0.030906$

$\frac{1}{X} = 36.769$

$\frac{1}{W} = 32.356$

$$\frac{1}{2}\,\frac{1.03905}{0.027197}\left[\frac{1.60858}{0.030906}-\frac{1.03905}{0.027197}\right]\left[1\pm\sqrt{1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}}\right]K^2$$

$$=\frac{0.029304-3.5416K^2}{0.030906}-\frac{0.039072}{0.027197}$$

$$1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}=\left[\frac{32.356}{19.1024\times 13.8424}\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{36.769\times 0.039072}{19.1024\times 13.8424}\frac{1}{K^2}-1\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left[0.12237\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{0.0054331}{K^2}-1\right]^2$$

$$=\left[\frac{0.0018472}{K^2}+0.56161\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left(\frac{0.0018472}{K^2}\right)^2+\frac{0.0020933}{K^2}+0.32105$$

$$\left(\frac{0.001}{K^2}\right)^2 3.41215+\left(\frac{0.001}{K^2}\right)6.0304-0.67895=0$$

$$\left(\frac{0.001}{K^2}\right)^2+1.7673\left(\frac{0.001}{K^2}\right)-0.19898=0$$

$$\left(\frac{0.001}{K^2}\right)=-0.88365+\sqrt{0.88365^2+0.19898}$$

$$=-0.88365+\sqrt{0.97982}=-0.88365+0.98986$$

$$=0.10621$$

$$\therefore \quad K^2=0.0094153 \qquad \boxed{k=0.097032}$$

$$\frac{\left(\frac{a}{R}\right)^2}{\left(\frac{t}{R}\right)}=\frac{1}{2}\,\frac{0.97032\times1.03905}{0.027197}\left[1\pm\sqrt{1-\frac{4\times3.9072\times0.027197\times1.0621}{1.03905^2}}\right]$$

$$=18.3845\times0.97032\times1.03905\left[1\mp\sqrt{0.58183}\right]$$

0.23722
1.76273

$$=4.397$$
32.674

If $\left(\frac{t}{R}\right)=\frac{1}{1000}$ $\qquad \left(\frac{a}{R}\right)^2=0.004397 \qquad \boxed{\frac{a}{R}=0.01135}$

~~$f\left(\frac{a}{R}\right)^2=0.1, \quad f=\frac{0.1}{0.032674}=3.060$~~

$$\frac{w_{max}}{t}=f\,\frac{\left(\frac{a}{R}\right)^4}{4}\,\frac{R}{t}=\frac{f\left(\frac{a}{R}\right)^2/(t/R)}{4}=\frac{0.1\times\overset{32.674}{4.397}}{4}=\overset{0.8169}{0.1099}$$

If we consider (ξg) also as a variable,

$$6q_2 + 4s_2 - \frac{1}{2} = 2\xi g\, p_2 + \frac{1}{3}\xi g$$

$$6q_2 - \frac{1}{2} = \xi g\, p_2 + 12\xi g\, r_2 - 5\xi g$$

$$-6q_2 - 2s_2 - \frac{1}{2} = 2\xi g\, p_2 + 6\xi g\, r_2 - \frac{5}{3}\xi g$$

$$-0.65 - 2.6q_2 - 4s_2 = 2.3\,\xi g\, p_2 + 1.2\,\xi g\, r_2 + 4.5\,\xi g$$

$$2.6q_2 - 0.65 = 1.95\,\xi g\, p_2 + 6.6\,\xi g\, r_2 - 0.2\,\xi g$$

This is a system of equations for 5 unknowns, $q_2, s_2, p_2, r_2, (\xi g)$

$$\begin{cases} q_2 + 0.666667\, s_2 - 0.33333\,\xi g\, p_2 + 0 = 0.055556\,\xi g + 0.083333 \\ q_2 + 0 - 0.166667\,\xi g\, p_2 - 2\xi g\, r_2 = -0.833333\,\xi g + 0.083333 \\ q_2 + 0.333333\, s_2 + 0.33333\,\xi g\, p_2 + \xi g\, r_2 = +0.277778\,\xi g - 0.083333 \\ q_2 + 1.53846\, s_2 + 0.88461\,\xi g\, p_2 + 0.46154\,\xi g\, r_2 = -1.73077\,\xi g - 0.25000 \\ q_2 + 0 - 0.75000\,\xi g\, p_2 - 2.53846\,\xi g\, r_2 = -0.076923\,\xi g + 0.25000 \end{cases}$$

$$0.666667\, s_2 - 0.166667\,\xi g\, p_2 + 2\,\xi g\, r_2 = 0.888889\,\xi g$$

$$0.33333\, s_2 + 0.5000\,\xi g\, p_2 + 3\xi g\, r_2 = 1.111111\,\xi g - 0.166667$$

$$1.20513\, s_2 + 0.55128\,\xi g \cdot p_2 - 0.53846\,\xi g\, r_2 = -2.00855\,\xi g - 0.16667$$

$$1.53846\, s_2 + 1.63461\,\xi g\, p_2 + 3.0000\,\xi g\, r_2 = -1.65385\,\xi g - 0.50000$$

$$S_2 - 0.25000\,\xi_g p_2 + 3.0000\,\xi_g r_2 = 1.33333\,\xi_g$$

$$S_2 + 1.5000\,\xi_g p_2 + 9.000\,\xi_g r_2 = 3.33333\,\xi_g - 0.50000$$

$$S_2 + 0.45744\,\xi_g p_2 - 0.44681\,\xi_g r_2 = -1.66667\,\xi_g - 0.138298$$

$$S_2 + 1.06250\,\xi_g p_2 + 1.950\,\xi_g r_2 = -1.07500\,\xi_g - 0.32500$$

$$1.75000\,\xi_g p_2 + 6.0000\,\xi_g r_2 = 2.00000\,\xi_g - 0.50000$$

$$1.04256\,\xi_g p_2 + 9.44681\,\xi_g r_2 = 5.0000\,\xi_g - 0.36170$$

$$0.60506\,\xi_g p_2 + 2.39681\,\xi_g r_2 = 0.59167\,\xi_g - 0.18170$$

$$\xi_g p_2 + 3.42857\,\xi_g r_2 = 1.14286\,\xi_g - 0.28571$$

$$\xi_g p_2 + 9.06116\,\xi_g r_2 = 4.79589\,\xi_g - 0.34693$$

$$\xi_g p_2 + 3.96128\,\xi_g r_2 = 0.97787\,\xi_g - 0.30856$$

$$5.63259\,\xi_g r_2 = 3.65303\,\xi_g - 0.06122$$

$$5.09988\,\xi_g r_2 = 3.81802\,\xi_g - 0.03837$$

$$4.21685\,\xi_g - 0.04238 = 3.65303\,\xi_g - 0.06122$$

$$\xi_g = -\frac{0.01884}{0.56382} = -0.03341$$

If compression is taken as positive

$$\cancel{\xi_g = 0.03341}$$

$$\boxed{\xi_g = 0.03341}$$

$$r_2 = \frac{-7.47105 \times 0.03341 - 0.09959}{-10.73247 \times 0.03341}$$

$$= \frac{0.09959 + 0.24961}{0.35857} = \boxed{0.97387 = r_2}$$

$$p_2 = \frac{-0.03341(-16.02115 + 6.91662) - 0.94120}{-3 \times 0.03341}$$

$$= -3.03484 + 9.39039 = \boxed{6.35555 = p_2}$$

$$s_2 = \frac{-0.03341(-17.60449 - 13.15035 + 1.92500) - 0.96330}{4}$$

$$= 0$$

$$q_2 = \frac{-0.03341(0.20376 + 2.99652 - 2.30768) + 0.083333}{5}$$

$$= \boxed{0.01070 = q_2}$$

Check $\quad 0.01070 + 0.33333 \times 0.03341 \times 6.3555$

$\quad = 0.08148 \qquad$ checks

$$q_0 = \frac{16}{3}g - 8 + \frac{1}{\xi g} = \frac{16}{3}g - 37.931$$

$$r_0 = \xi g\left(\frac{2}{3}g - \frac{4}{3}\right)$$

$$q_2 = 0.01070$$

$$s_2 = 0$$

$$p_2 = 6.3555$$

$$r_2 = 0.97347$$

$$\xi g = -0.03341$$

$$\xi g = -0.03341$$

$$= \frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g$$

$$\left(\frac{a}{R}\right)^2 = -\frac{64 \times 0.03341}{g}\left(\frac{\sigma}{E}\right)$$

$$\frac{\tilde{\mathcal{E}}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{\frac{4}{9}(\xi g)^2\left(g^2 - 4g + 4\right) + 12 \times 0.01070^2\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\left(g^2 - 4g + 4\right) + 0.0013379\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\,g^2 - 0.0019844\,g + 0.0033223\right\}\right]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[0.0012899\,g^2 - 0.0051594\,g + 0.0086380\right]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.0334)^2\Big[0.35(5.3333g - 37.931)^2 + (3.7333 - 2.8000g) \times (5.3333g - 37.931)$$

$$+ 15.6444 - 31.2889g + 34.5016g^2 + 8.9125$$

$$+ 8.3 \times 6.3555^2 + 25.2 \times 6.3555 \times 0.97367 - 5.3 \times 6.3555 + 22.05 \times 0.97367 + 7.4 \times 0.97367^2\Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.03341)^2\Big[$$

g^2	g	
+ 9.95555	− 141.609	+ 503.536
− 14.93333	+ 106.207	− 141.609
+ 34.5016	+ 19.911	+ 15.644
29.1238	− 31.289	+ 8.913
	− 46.780	+ 335.757
		+ 155.974
		− 33.684
		+ 7.018
		+ 21.474
		872.553

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.032508g^2 - 0.052216g + 0.97394\Big]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.1226\frac{g^2}{\left(\frac{\sigma}{E}\frac{R}{t}\right)^2}\Big]$$

$$\frac{\delta\mathcal{E}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[+0.013367g - 0.026733\Big] \times 2 \quad ?$$

$$0.033798\,g^2 - 0.084109\,g + 0.03604 + \frac{0.12210\,g^2}{K^2} = 0$$

$$g^2 - 2.4886\,g + 1.0663 = 0$$

$$g = 1.2443 \pm \sqrt{1.2443^2 - 1.0663} = 1.2443 \pm \sqrt{0.4820} = 1.2443 \pm 0.6943$$

$$= 0.5500$$

$$1.9386$$

$$K^2 = \frac{g^2}{0.68885\,g - (0.29517 + 0.27681\,g^2)}$$

①	②	③	④	⑤	⑥	⑦	⑧
g	g^2	$0.68885g$	$0.27681g^2$	③ - (0.29517 + ④)	③/⑤ = K^2		
0.60	0.36	0.41331	0.09965	0.01849	19.47		
0.70	0.49	0.48220	0.13564	0.05139	9.53		
0.80	0.64	0.55108	0.17716	0.07875	8.13		
0.90	0.81	0.61997	0.22422	0.10058	8.05		
1.00	1.00	0.68885	0.27681	0.11687	8.55		
1.10	1.21	0.75774	0.33494	0.12763	9.49		
1.20	1.44	0.82662	0.39861	0.13284	10.84		
1.40	1.69	0.96439	0.46781	0.20141			
1.60	2.56	1.10216	0.70863	0.09836			
1.80	3.24	1.23993	0.89686	0.04790			

Comparison of different energies

For $g = 0.90$,

(Difference!)

$\frac{\mathcal{E}_1}{R^3}$ = extensional strain energy outside the circular region

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\begin{array}{r} 0.001044 \\ -0.004640 \\ 0.008638 \\ \hline 0.005042 \end{array}\right]$$

$$\xi g = -0.03341$$

$$\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g = -0.03341$$

$$\left(\frac{a}{R}\right)^2 = \frac{-64 \times 0.03341}{g}\frac{\sigma}{E}$$

For the circular region:

$$\widehat{rr} = \frac{Ef}{64}\left[\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2P_2 + \frac{1}{3}\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\left[\left\{\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}f\left(\frac{r}{R}\right)^6\right\} + \cos 2\theta\left\{2P_2 + 12R_2\left(\frac{r}{R}\right)^2 - 5\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\widehat{r\theta} = \frac{Ef}{64}\left[\sin 2\theta\left\{2P_2 + 6R_2\left(\frac{r}{R}\right)^2 - \frac{5}{3}\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{1}{E}\left(\widehat{rr} - \nu\widehat{\theta\theta}\right) = \frac{f}{64}\left[\left\{(1-\nu)\frac{Q_0}{2} + 4(1-3\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}(1-5\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(1-7\nu)f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2(1+\nu)P_2 + 12\nu R_2\left(\frac{r}{R}\right)^2 + \left(\frac{1}{3} - 5\nu\right)\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_0}{\partial r}\right)^2\right\} = \frac{f}{64}\left\{32\left(\frac{a}{R}\right)^2\left(f\frac{a^2}{R^2} - 1\right)\frac{r^2}{R^2} - 32\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 + 32f\left(\frac{r}{R}\right)^6 + \cos 2\theta\left(32\left(\frac{a}{R}\right)^2 - 32\left(\frac{r}{R}\right)^2\right)\left(\frac{r}{R}\right)^2\right\}$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{f}{64}\left[\left\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2(1+\nu)P_2 + 4\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{r}{R}\right)^2 - 5\left(\frac{19}{3} + \nu\right)\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2}Q_0\left(\frac{a}{R}\right) + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^3 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^5 - \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^7\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2\left(\frac{a}{R}\right) + \frac{4}{3}\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{a}{R}\right)^3 - \left(\frac{19}{3}+\nu\right)\left(\frac{a}{R}\right)^5\Big\}\Big]$$

$$\frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2}Q_0 + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^6\Big\} + \cos 2\theta\Big\{2(1+\nu)P_2 + 12R_2\left(\frac{a}{R}\right)^2 - \left(5 - \frac{1}{3}\nu\right)\left(\frac{a}{R}\right)^4\Big\}\Big]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{f}{64}\cos 2\theta\Big\{4(1+\nu)P_2 + \frac{32}{3}\left(\frac{a}{R}\right)^2\left(\frac{a}{R}\right)^2 + 12(1+\nu)R_2\left(\frac{a}{R}\right)^2 - \frac{2}{3}(17+\nu)\left(\frac{a}{R}\right)^4\Big\}$$

$$\frac{v}{R} = \frac{f}{64}\sin 2\theta\Big\{2(1+\nu)P_2\left(\frac{a}{R}\right) + \frac{16}{3}\left(\frac{a}{R}\right)^2\left(\frac{a}{R}\right)^3 + 6(1+\nu)R_2\left(\frac{a}{R}\right)^3 - \frac{1}{3}(17+\nu)\left(\frac{a}{R}\right)^5\Big\}$$

For the region outside the circle:

$$\sigma_r = \sigma\Big[\frac{1}{2} + \frac{P_0}{\left(\frac{a}{R}\right)^2} + \cos 2\theta\Big\{\frac{1}{2} - \frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{4S_2}{\left(\frac{a}{R}\right)^2}\Big\}\Big]$$

$$\sigma_\theta = \sigma\Big[\frac{1}{2} - \frac{P_0}{\left(\frac{a}{R}\right)^2} + \cos 2\theta\Big\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\Big\}\Big]$$

$$\tau_{r\theta} = -\sigma\sin 2\theta\Big\{\frac{1}{2} + \frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{2S_2}{\left(\frac{a}{R}\right)^2}\Big\}$$

$$\frac{u}{R} = \frac{\sigma}{E}\left[\frac{1}{2}(1-\nu)\left(\frac{r}{R}\right) - (1+\nu)\frac{R_0}{\left(\frac{r}{R}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{R}\right) + 2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} + \frac{4S_2}{\left(\frac{r}{R}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\sin 2\theta\left\{2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{R}\right)\right\}$$

With the simplified relation of p. 316, the condition of stress continuity becomes

$$\frac{1}{2} + r_0 = \xi g\left\{\frac{1}{2}q_0 + 4(1-g) + \frac{4}{3}(2g-1) - \frac{2}{3}g\right\}$$

$$-\frac{1}{2} + 6q_2 + 4S_2 = \xi g\left\{2p_2 + \frac{1}{3}\right\}$$

$$\frac{1}{2} - r_0 = \xi g\left\{\frac{1}{2}q_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{14}{3}g\right\}$$

$$6q_2 - \frac{1}{2} = \xi g\left\{2p_2 + 12r_2 - 5\right\}$$

$$-\frac{1}{2} - 6q_2 - 2S_2 = \xi g\left\{2p_2 + 6r_2 - \frac{5}{3}\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4S_2 = \xi g\left\{-2(1+\nu)p_2 - \frac{4}{3}(8+30r_2) + \left(\frac{19}{3}+\nu\right)\right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \xi g\left\{2(1+\nu)p_2 + 6(1+\nu)r_2\ \frac{1}{3}(1+\nu)\right\}$$

Thus

$$q_0 = \frac{1}{g\xi} - 8\left(1-\frac{2}{3}g\right)$$

$$r_0 = \xi g\,\frac{2}{3}(g-2).$$

$$q_2 + 0.666667\, s_2 - 0.083333 = \xi g \{ 0.33333\, p_2 + 0.055556 \}$$

$$q_2 + 0 \quad - 0.083333 = \xi g \{ 0.33333\, p_2 + 2 r_2 - 0.833333 \}$$

$$-q_2 - 0.33333\, s_2 - 0.083333 = \xi g \{ 0.33333\, p_2 + r_2 - 0.277778 \}$$

$$q_2 + 1.53846\, s_2 + 0.25000 = \xi g \{ - p_2 - 0.461538\, r_2 - 1.55128 \}$$

$$q_2 \quad - 0.25000 = \xi g \{ p_2 + 3 r_2 - 0.155128 \}$$

$$0.666667\, s_2 + 0 = \xi g \{ - 2 r_2 + 0.888889 \}$$

$$- 0.33333\, s_2 - 0.1666667 = \xi g \{ 0.66667\, p_2 + 3 r_2 - 1.111111 \}$$

$$1.20513\, s_2 + 0.166667 = \xi g \{ -0.66667\, p_2 + 0.538462\, r_2 - 1.82906 \}$$

$$1.53846\, s_2 + 0.50000 = \xi g \{ - 2 p_2 - 3.461538\, r_2 - 1.39615 \}$$

$$s_2 + 0 = \xi g \{ - 3 r_2 + 1.33333 \}$$

$$- s_2 - 0.5000 = \xi g \{ 2 p_2 + 9 r_2 - 3.33333 \}$$

$$s_2 + 0.138298 = \xi g \{ -0.553191\, p_2 + 0.446808\, r_2 - 1.51773 \}$$

$$s_2 + 0.325 = \xi g \{ - 1.3 p_2 - 2.25000\, r_2 - 0.907498 \}$$

$$2 \xi g\, p_2 + 6 \xi g\, r_2 = 2 \xi g - 0.5000$$

$$1.446809\, \xi g\, p_2 + 9.446808\, \xi g\, r_2 = 4.85106\, \xi g - 0.361702$$

$$0.74681\, \xi g\, p_2 + 2.696608\, \xi g\, r_2 = 0.61023\, \xi g - 0.186702$$

$$\xi_g p_2 + 3\xi_g r_2 = \xi_g - 0.250000$$

$$\xi_g p_2 + 6.52941\,\xi_g r_2 = 3.35294\,\xi_g - 0.25000$$

$$\xi_g p_2 + 3.61111\,\xi_g r_2 = 0.817116\,\xi_g - 0.25000$$

$$\left.\begin{aligned} 3.52941\,\xi_g r_2 &= 2.35294\,\xi_g \\ 2.91830\,\xi_g r_2 &= 2.53562\,\xi_g \end{aligned}\right\} \text{ Impossible}$$

Method of Least Square:

$$q_2 + 0.507692\,s_2 - 0.0166667 = \xi_g\{0.0666667\,p_2 + 0.707692\,r_2 - 0.441281\}$$

$$1.5000\,q_2 + s_2 - 0.12500 = \xi_g\{0.50000\,p_2 + 0.083333\}$$

$$3\,q_2 + s_2 + 0.25000 = \xi_g\{-p_2 - 3r_2 + 0.833333\}$$

$$0.65\,q_2 + s_2 + 0.1625 = \xi_g\{-0.65\,p_2 - 0.3000\,r_2 - 1.00833\}$$

$$q_2 + 0.582525\,s_2 + 0.0558253 = \xi_g\{-0.223301\,p_2 - 0.640778\,r_2 - 0.0177994\}$$

$$0.666667\, q_2 + 0.444444\, s_2 - 0.055556 = \xi\frac{q}{l}\left\{0.222222\, p_2 + 0.0370371\right\}$$

$$0.333333\, q_2 + 0.1111111\, s_2 + 0.0277777 = \xi\frac{q}{l}\left\{-0.111111\, p_2 - 0.33333\, r_2 + 0.0925926\right\}$$

$$1.53846\, q_2 + 2.36686\, s_2 + 0.384615 = \xi\frac{q}{l}\left\{-1.53846\, p_2 - 0.710058\, r_2 - 2.38658\right\}$$

$$2.53846\, q_2 + 2.92241\, s_2 + 0.356836 = \xi\frac{q}{l}\left\{-1.42735\, p_2 - 1.043391\, r_2 - 2.25695\right\}$$

$$\boxed{q_2 + 1.15125\, s_2 + 0.140573 = \xi\frac{q}{l}\left\{-0.562289\, p_2 - 0.411032\, r_2 - 0.889101\right\}}$$

$$0.333333\, q_2 + 0.222222\, s_2 - 0.0277777 = \xi\frac{q}{l}\left\{0.111111\, p_2 + 0.0185185\right\}$$

$$0.33333\, q_2 \qquad - 0.0277777 = \xi\frac{q}{l}\left\{0.11111\, p_2 + 0.666667\, r_2 - 0.277777\right\}$$

$$-0.33333\, q_2 - 0.11111\, s_2 - 0.0277777 = \xi\frac{q}{l}\left\{0.11111\, p_2 + 0.33333\, r_2 - 0.0925926\right\}$$

$$-\, q_2 - 1.53846\, s_2 - 0.25000 = \xi\frac{q}{l}\left\{p_2 + 0.461538\, r_2 + 1.55128\right\}$$

$$q_2 \qquad - 0.25000 = \xi\frac{q}{l}\left\{p_2 + 3\, r_2 - 0.155128\right\}$$

$$0.33333\, q_2 - 1.42735\, s_2 - 0.583333 = \xi\frac{q}{l}\left\{2.33333\, p_2 + 4.461538\, r_2 + 1.04430\right\}$$

$$\boxed{q_2 - 4.28205\, s_2 - 1.75000 = \xi\frac{q}{l}\left\{7\, p_2 + 13.3846\, r_2 + 3.13290\right\}}$$

$$2q_2 + 0 - 0.1666667 = \xi^q_q\{0.666667\,p_2 + 7\,r_2 - 1.666667\}$$

$$-q_2 - 0.33333\,s_2 - 0.083333 = \xi^q_q\{0.33333\,p_2 + r_2 - 0.277778\}$$

$$-0.461538\,q_2 - 0.710058\,s_2 - 0.115385 = \xi^q_q\{0.461538\,p_2 + 0.213017\,r_2 + 0.715975\}$$

$$3q_2 \qquad - 0.25000 = \xi^q_q\{3\,p_2 + 9\,r_2 - 0.465384\}$$

$$3.538462\,q_2 - 1.043391\,s_2 - 1.115385 = \xi^q_q\{4.461538\,p_2 + 14.213017\,r_2 - 1.693857\}$$

$$\boxed{q_2 - 0.294871\,s_2 - 0.315218 = \xi^q_q\{1.26087\,p_2 + 4.01672\,r_2 - 0.478698\}}$$

The equations for constants are then

$$q_2 + 0.507692\, s_2 - 0.016667\, \xi g\, p_2 - 0.707692\, \xi g\, r_2 = -0.441281\, \xi g + 0.016667$$
$$q_2 + 1.15125\, s_2 + 0.562289\, \xi g\, p_2 + 0.411039\, \xi g\, r_2 = -0.889131\, \xi g - 0.140573$$
$$q_2 - 4.28205\, s_2 - 7.000000\, \xi g\, p_2 - 13.3846\, \xi g\, r_2 = 3.13290\, \xi g + 1.750000$$
$$q_2 - 0.294871\, s_2 - 1.26087\, \xi g\, p_2 - 4.01672\, \xi g\, r_2 = -0.478698\, \xi g + 0.315218$$

$$0.64356\, s_2 + 0.628956\, \xi g\, p_2 + 1.118724\, \xi g\, r_2 = -0.447820\, \xi g - 0.157240$$
$$5.43330\, s_2 + 7.562289\, \xi g\, p_2 + 13.79563\, \xi g\, r_2 = -4.02200\, \xi g - 1.890573$$
$$3.98718\, s_2 + 5.73913\, \xi g\, p_2 + 9.36788\, \xi g\, r_2 = -3.61160\, \xi g - 1.43478$$

$$s_2 + 0.977310\, \xi g\, p_2 + 1.73834\, \xi g\, r_2 = -0.695850\, \xi g - 0.244329$$
$$s_2 + 1.391839\, \xi g\, p_2 + 2.53909\, \xi g\, r_2 = -0.740249\, \xi g - 0.347960$$
$$s_2 + 1.439397\, \xi g\, p_2 + 2.34950\, \xi g\, r_2 = -0.905804\, \xi g - 0.359849$$

$$0.462087\, \xi g\, p_2 + 0.61116\, \xi g\, r_2 = -0.209954\, \xi g - 0.115520$$
$$0.414529\, \xi g\, p_2 + 0.80075\, \xi g\, r_2 = -0.044399\, \xi g - 0.103631$$

$$\xi g\, p_2 + 1.32261\, \xi g\, r_2 = -0.454359\, \xi g - 0.249996$$
$$\xi g\, p_2 + 1.93171\, \xi g\, r_2 = -0.107107\, \xi g - 0.249996$$

$$0.60910\, \xi g\, r_2 = 0.347252\, \xi g$$

$$\xi g \kappa_2 = 0.570106\, \xi g$$

$$2\, \xi g \ell_2 = -2.41678\, \xi g - 2 \times 0.25000$$

$$\xi g \ell_2 = -1.20839\, \xi g - 0.25000$$

$$3 s_2 = (4.60233 - 3.77805 - 2.34190)\, \xi g + 0.952163 - 0.952138$$

$$s_2 = -0.505873\, \xi g$$

$$4 q_2 = \xi g (-1.47613 - 9.38345 + 10.08972 + 1.32382) - 1.94131 + 1.94131$$

$$q_2 = 0.138490\, \xi g$$

Check: ξg

+0.138490	
−0.256828	
+0.080559	+ 0.0166667
−0.403459	
−0.441238	

O. K.

The extensional strain energy in the circular region

$$\hat{\varepsilon}_r+\hat{\varepsilon}_\theta = \frac{Ef}{64}\left[\left\{Q_0+16\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2+8\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4-\frac{16}{3}f\left(\frac{a}{R}\right)^6\right\}\right.$$

$$\left.+\cos 2\theta\left\{12R_2\left(\frac{a}{R}\right)^2-\frac{16}{3}\left(\frac{a}{R}\right)^4\right\}\right]$$

$$-\left\{2P_2+\frac{1}{3}\left(\frac{a}{R}\right)^4\right\}\left\{2P_2+12R_2\left(\frac{a}{R}\right)^2-5\left(\frac{a}{R}\right)^4\right\}-\left\{2P_2+6R_2\left(\frac{a}{R}\right)^2-\frac{5}{3}\left(\frac{a}{R}\right)^4\right\}^2$$

$$=-\left[4P_2^2+\frac{2}{3}P_2\left(\frac{a}{R}\right)^4+24P_2R_2\left(\frac{a}{R}\right)^2+4R_2\left(\frac{a}{R}\right)^6-10P_2\left(\frac{a}{R}\right)^4-\frac{5}{3}\left(\frac{a}{R}\right)^6\right]$$

$$-\left[4P_2^2+36R_2^2\left(\frac{a}{R}\right)^4+\frac{25}{9}\left(\frac{a}{R}\right)^8+24P_2R_2\left(\frac{a}{R}\right)^2-\frac{20}{3}P_2\left(\frac{a}{R}\right)^4-20R_2\left(\frac{a}{R}\right)^6\right]$$

$$=-\left[8P_2^2+36R_2^2\left(\frac{a}{R}\right)^4+48P_2R_2\left(\frac{a}{R}\right)^2-16P_2\left(\frac{a}{R}\right)^4-16R_2\left(\frac{a}{R}\right)^6+\frac{10}{9}\left(\frac{a}{R}\right)^8\right]$$

$$+2(1+\nu)\left[4P_2^2\left(\frac{a}{R}\right)^2+6R_2^2\left(\frac{a}{R}\right)^6+12P_2R_2\left(\frac{a}{R}\right)^4-\frac{8}{3}P_2\left(\frac{a}{R}\right)^6-2R_2\left(\frac{a}{R}\right)^8+\frac{1}{9}\left(\frac{a}{R}\right)^{10}\right]$$

$$+\left[24R_2^2\left(\frac{a}{R}\right)^6-16R_2\left(\frac{a}{R}\right)^8+\frac{16\times 1.6}{3}\left(\frac{a}{R}\right)^{10}\right]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[s_2^2 + 2(1+\nu)\left\{(r_0^2 + 2s_2^2) + 12 s_2 p_2 + 12 p_2^2\right\}\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2\Big[\left\{p_0^2 + \frac{8}{3}(4-3g)p_0 + 34.1333 - 49.7778g + 26.4127g^2\right\}$$

$$-(1+\nu)\left\{\frac{1}{2}p_0^2 + 4\left(\frac{4}{3} - g\right)p_0 + 14.2222 - 14.2222g - 6.2222g^2\right\}$$

$$+\left\{24 r_2^2 - 16 r_2 + \frac{25.6}{3}\right\}$$

$$+(1+\nu)\left\{8p_2^2 + 12 r_2^2 + 24 p_2 r_2 - \frac{16}{3}p_2 - 4r_2 + \frac{2}{9}\right\}\Big]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \; 0.122100\,\frac{g^2}{K^2}$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{2(1-\nu)s_2 - 4\nu r_0\right\}$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.50587^2\xi^2g^2 + 2.6\Big\{\xi^2g^2\frac{4}{9}(g^2 - 4g + 4) + 2\times 0.50587^2\xi^2g^2$$

$$-12\times 0.13849\times 0.50587\,\xi^2g^2 + 12\times 0.13849^2\xi^2g^2\Big\}\Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\xi^2g^2\Big(\begin{matrix}1.58660\\1.77777\\-0.84070\\+0.23015\end{matrix} - 1.77777g + 0.44444g^2\Big)\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\xi^2g^2\left(0.44444g^2 - 1.77776g + 2.75383\right)\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2 \Big[\Big\{0.35\, g_0^{\,2} + (3.7333 - 2.80000 g)\, g_0$$

$$+ 15.6445 - 31.2890 g + 34.5015 g^2\Big\}$$

$$+ \Big\{10.4\, p_2^{\,2} + 39.6\, r_2^{\,2} + 31.2\, p_2 r_2 - 6.93333\, p_2 - 21.2\, r_2 + 8.82222\Big\}\Big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \Big[\, 0.35\big(1 - 8(\xi g) + 5.33333(\xi g) g\big)^2$$

$$+ \xi g\,(3.7333 - 2.80000 g)\big(1 - 8(\xi g) + 5.3333(\xi g) g\big)$$

$$+ (15.6445 - 31.2890 g + 34.5015 g^2)\,\xi^2 g^2$$

$$+ 10.4 \times (1.20839\,\xi g + 0.25000)^2 + 39.6 \times 0.570106\,\xi^2 g^2 - 31.2 \times 0.570106\,\xi g \times (1.20839\,\xi g + 0.25000)$$

$$+ 6.93333\,\xi g\,(1.20839\,\xi g + 0.25000) - 21.2\,\xi g \times 0.570106\,\xi g + 8.82222\,\xi^2 g^2\Big\}\Big]$$

$\xi^2 g^2 \Big[$

g^2	g	
9.95555	− 29.8666	+ 22.40000
− 14.93333	+ 42.3111	− 29.86666
+ 34.5015	− 31.2890	+ 15.6445
		+ 15.1862
		+ 22.5762
		− 21.4940
		+ 8.3782
		− 12.0862
		+ 8.8222

$$(\xi g)\Big[\; g \;\; \begin{matrix} 3.73333 \\ -2.80000 \end{matrix} \qquad \begin{matrix} -\;5.6 \\ +\;3.73333 \\ +\;6.28363 \\ -\;4.44683 \\ +\;1.73333 \end{matrix} \;\Big]$$

$$+\big[\; 0.35 + 0.65 \;\big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\;\xi^2 g^2\left(29.5237\, g^2 - 18.8445 + 29.5605\right) + \xi g\left(0.93333\, g + 1.70346\right)\Big]$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a^2}{R^2}\right)\Big\{-1.4\times 0.50587\,(\xi g) - 1.2\times 0.66167\,(g-2)\,\xi g\Big\}$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{0.89178 - 0.80000\, g\Big\}(\xi g)$$

$$\frac{\mathcal{C}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{\;\xi^2 g^2\left(29.9681\, g^2 - 20.6223\, g + 32.3143\right) + \xi g\left(1.73333\, g + 0.81168\right) + 0.122100\,\frac{g^2}{K^2}\Big\}$$

Let σ be compression,

$$K^2 = \frac{0.122100\, g}{\xi\,(1.73333\, g + 0.81168) - \xi^2 (29.9681\, g^3 - 20.6223\, g^2 + 32.3143\, g)}$$

$$\xi = \frac{1}{2}\,\frac{1.73333\, g + 0.81168}{g\,(29.9681\, g^2 - 20.6223\, g + 32.3143)} = \frac{1}{64}\,\frac{1}{\left(\frac{t}{R}\right)}\,\frac{1}{K}\left(\frac{a}{R}\right)^2$$

$$K^2 = \frac{0.488400\, g^2 (29.9681\, g^2 - 20.6223\, g + 32.3143)}{(1.7333\, g + 0.81168)^2}$$

g = amplitude factor

When $g = \frac{0.89178}{0.8000} = 1.1147$

$$K^2 = \frac{0.4884 \times 1.2426 \times (37.238 - 22.988 + 32.3143)}{(0.93335}$$

$g = 0.1$

$$K = 0.1\,\frac{\sqrt{0.488400\,(30.5518)}}{0.98501} = \underline{\underline{0.3920}}$$

$$\left(\frac{a}{R}\right)^2 = 32\left(\frac{t}{R}\right)\frac{\sqrt{0.488400}}{\sqrt{29.9681\, g^2 - 20.6223\, g + 32.3143}} = 4.04\left(\frac{t}{R}\right)$$

$\left(\frac{t}{R}\right) = \frac{1}{1000}$, $\frac{a}{R} = 0.0636$

at $g = 0.1$

$$\xi^2 g^2 = \frac{1}{4} \frac{(1.73333g + 0.81168)^2}{(29.9681g^2 - 20.6223g + 32.3143)^2} = 0.00025985$$

$$\xi g = \frac{1}{2} \frac{1.7333g + 0.81168}{29.9681g^2 - 20.6223g + 32.3143} = \frac{1}{2} \frac{0.98501}{30.5518} = 0.016120$$

$$\mathcal{E}_1 \sim 0.00025985 \times 2.58049 = \underline{0.00067054} \; +$$ Energy outside the circular region

$$\mathcal{E}_2 \sim 0.00025985 \times 27.9713 - 0.016120 \times 1.79679$$

$$= 0.007268 - 0.028964 = \underline{-0.021696}$$ Energy – extensional in the region

$$\mathcal{E}_3 \sim 0.122100 \frac{0.01}{0.3920^2} = \underline{+0.00795}$$ Bending Energy

$$\wp \sim -0.016120 \times 0.81178 = \underline{-0.013086} \; (?)$$ Increase in potential

$$\frac{w_{max}}{t} = f \frac{1}{4} \left(\frac{a}{R}\right)^4 \left(\frac{R}{t}\right) = \frac{g}{4} \left(\frac{a}{R}\right)^2 \left(\frac{R}{t}\right) = \frac{4.04}{4} g = 0.101 \quad \text{(Too small)}$$

by virtual work

$$\frac{\delta P}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-2(1+\nu)\Lambda_3 + 4\delta_2\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{2}\right)^2\left\{-\xi g \times 26 \times 0.666667(g-2) - 2.02349\,\xi g\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{1.44318 - 1.73333\,g\right\}\xi g$$

$$k^2 = \frac{0.4884\,g\,(29.9681\,g^2 - 20.6223\,g + 32.3143)}{(2.6666\,g + 0.26028)^2}$$

$k = 1.2[illegible]$

$g = 0.1$ $\qquad k = \frac{0.0699\sqrt{30.55}}{0.52[illegible]} = 0.[illegible]$

$g = 0.05$ $\qquad k = \frac{0.03495}{0.39361}\sqrt{31.36} = 0.498$

$g = 0.15$ $\qquad k = \frac{0.1048\sqrt{29.914}}{0.6598} = 0.870$

~~Take $2.91830\,\xi g\, r_2 = 2.53582\,\xi g$~~

~~or $\xi g\, r_2 = 0.86894\,\xi g$~~

If we drop the condition of continuity of u,

$$q_2 + 0.66667\, s_2 - 0.083333 = \xi g\left\{0.33333\, p_2 + 0.055556\right\}$$

$$q_2 \qquad - 0.083333 = \xi g\left\{0.3333\, p_2 + 2 r_2 - 0.33333\right\}$$

$$-q_2 - 0.33333\, s_2 - 0.083333 = \xi g\left\{0.33333\, p_2 + r_2 - 0.277778\right\}$$

$$q_2 \qquad - 0.250000 = \xi g\left\{p_2 + 3 r_2 - 0.155198\right\}$$

$$0.666667\, s_2 + 0 = \xi g\left\{-2 r_2 + 0.888889\right\}$$

$$-0.33333\, s_2 - 0.166667 = \xi g\left\{0.66667\, p_2 + 3 r_2 - 1.111111\right\}$$

$$-0.33333\, s_2 - 0.33333 = \xi g\left\{1.3333\, p_2 + 4 r_2 - 0.432103\right\}$$

$$s_2 + 0 = \xi g\left\{-3 r_2 + 1.33333\right\}$$

$$-s_2 - 0.5000 = \xi g\left\{2 p_2 + 9 r_2 - 3.33333\right\}$$

$$-s_2 - 1.000 = \xi g\left\{4 p_2 + 12 r_2 - 1.296309\right\}$$

$$2\xi g p_2 + 6\,\xi g r_2 = 2\,\xi g - 0.50000$$

$$2\xi g p_2 + 3\,\xi g r_2 = -2.03462\,\xi g - 0.5000$$

$$3\,\xi g r_2 = 4.03462\,\xi g$$

$$\xi g r_2 = 1.34487\, \xi g$$

$$\xi g p_2 = \frac{1}{4}\left\{(-12.1038 - 0.03462)\xi g - 1.000\right\}$$

$$\xi g p_2 = -3.034675\, \xi g - 0.2500$$

$$3 s_2 + 1.5000 = \xi g\left\{-6 p_2 - 24 r_2 + 5.96537\right\}$$

$$= \xi g\left\{18.2081 - 32.27688 + 5.96537\right\} + 1.5000$$

$$s_2 = -2.70113\, \xi g$$

$$4 q_2 + s_2 - 0.33333 = \xi g\left\{1.3333\, p_2 + 4 r_2 - 0.655127\right\}$$

$$q_2 = \frac{1}{4}\left\{2.70113 - 4.04623 + 5.37948 - 0.65513\right\}\xi g$$

$$q_2 = 0.84481\, \xi g$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,2.70113^2 + 2.6\Big\{\frac{4}{9}(g^2 - 4g + 4) + 2\times 2.70113^2$$

$$- 12\times 2.70113\times 0.84481 + 12\times 0.84481^2\Big\}\Big](\xi g)^2$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\;0.44444\,g^2 - 1.77778\,g + \begin{array}{r} 1.77778 \\ 7.29610 \\ 37.93972 \\ -\,27.38330 \\ +\,8.56445 \end{array}\right](\xi g)^2$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\;0.44444\,g^2 - 1.77778\,g + 28.19475\Big](\xi g)^2$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,0.35\big(1 - 8\,\xi g + 5.3333(\xi g)^2\big) + \xi g\,(3.7333 - 2.8000\,\xi g)(1 - 8\,\xi g + 5.333\,\xi g)$$

$$+ (34.5015\,g^2 - 31.2880\,g + 15.6445)\,\xi g\big)^2$$

$$+ 10.4\,(3.03475\,\xi g + 0.2500)^2 + 39.6\times 1.34467^2(\xi g)^2$$

$$- \quad 35.02661\,\xi g\,(3.034675\,\xi g + 0.2500) - 28.51124\,(\xi g)^2 + 8.8222\,(\xi g)^2\Big]$$

$$(\xi g)^2\left[\;29.5237\,g^2 - 18.8445\,g + \begin{array}{r} 22.40000 \\ -\,29.86666 \\ +\,15.6445 \\ +\,95.7810 \\ +\,71.6237 \\ -\,106.2944 \\ -\,28.5112 \\ +\,8.8222 \end{array}\right]$$

$$\xi g \Big[\begin{matrix} 3.73333\, g \\ -2.80000 \end{matrix} \qquad \begin{matrix} -5.6 \\ 3.73333 \\ +15.7807 \\ -8.7567 \end{matrix} \Big]$$

$$\frac{\mathscr{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \Big[(\xi g)^2 (29.5237 g^2 - 18.8445 g + 49.5991)$$

$$+ \xi g\, (0.93333\, g + 5.1573) \Big]$$

$$\frac{\mathscr{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \Big\{ - \xi g \times 2.6 \times 0.66667\, (g-2) - 10.8045\, \xi g \Big\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 (\xi g) \Big\{ -1.73333\, g - 7.3378 \Big\}$$

$$\left(\frac{\omega}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$\beta = 3.8317$

$\eta = 0.4028$

$\delta = 1.4028$

$$\left(\frac{\omega}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left(\frac{1}{2}\frac{a^2}{R^2}\right)\left\{J_0\left(\beta\frac{r}{a}\right) + \eta\right\}\frac{1}{\delta}$$

$$\frac{1}{R}\frac{\partial\omega}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta}{a}J_0'\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial\omega_0}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta$$

$$\frac{1}{R}\frac{\partial^2\omega}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta^2}{a^2}J_0''\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial^2\omega_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta$$

$$-\left\{\frac{1}{r}\frac{\partial\omega}{\partial r}\frac{\partial^2\omega}{\partial r^2} - \frac{1}{r}\frac{\partial\omega_0}{\partial r}\frac{\partial^2\omega_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\frac{a\beta}{r}J_0'\left(\beta\frac{r}{a}\right)\right]\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''\left(\beta\frac{r}{a}\right)\right]$$

$$= -\frac{1}{R^2}\left[\frac{1}{2}\frac{f}{\delta}\beta^2\sin^2\theta\left\{J_0'' + \frac{1}{\left(\beta\frac{r}{a}\right)}J_0'\right\} + \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r}J_0'J_0''\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2\omega}{\partial r^2}\frac{\partial^2\omega}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2\omega_0}{\partial r^2}\frac{\partial^2\omega_0}{\partial\theta^2}\right\}$$

$$= -\frac{1}{R^2}\cos 2\theta \cdot \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''$$

$$\nabla^4\phi = \frac{E}{R^2}\left[J_0 \frac{1}{2}\frac{f}{\delta}\beta^2 \sin^2\theta - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta\right] \quad \underline{\underline{362}}$$

$$= \frac{E}{R^2}\left[J_0 \frac{1}{4}\frac{f}{\delta}\beta^2(1-\cos 2\theta) - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta\right]$$

$$= \frac{1}{4}\frac{f}{\delta}\beta^2\frac{E}{R^2}\left[\left\{J_0 - \beta^2\frac{f}{\delta}\frac{J_0' J_0''}{(\beta\frac{r}{a})}\right\} - \cos 2\theta\left\{J_0 + 2J_0''\right\}\right]$$

$$= \frac{1}{4}\frac{gE}{R^2}\left[\left\{J_0 - g\frac{J_0' J_0''}{z}\right\} - \cos 2\theta\left(J_0 + 2J_0''\right)\right\}$$

where $g = \frac{f}{\delta}\beta^2$, $z = (\beta\frac{r}{a})$

$$\frac{J_0' J_0''}{z} = \frac{J_1 J_1'}{z} = \frac{J_1^2}{z^2} - \frac{J_1 J_2}{z}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r (2r+2)!\,(\frac{1}{2}z)^{2r}}{4(r!)(r+1)!\,(r+1)!\,(r+2)!} - \sum_{r=0}^{\infty}\frac{(-1)^r (2r+3)!\,(\frac{1}{2}z)^{2r+2}}{2(r!)(r+1)!\,(r+2)!\,(r+3)!}$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{z}{2}\right)^2 + \frac{5}{48}\left(\frac{z}{2}\right)^4 - \frac{7}{288}\left(\frac{z}{2}\right)^6 + \frac{7}{1920}\left(\frac{z}{2}\right)^8 - \frac{11}{28800}\left(\frac{z}{2}\right)^{10}$$

$$+ \frac{143}{4838400}\left(\frac{z}{2}\right)^{12} - \cdots\cdots$$

$$-\left\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{24}\left(\frac{z}{2}\right)^4 + \frac{7}{96}\left(\frac{z}{2}\right)^6 - \frac{7}{480}\left(\frac{z}{2}\right)^8 + \frac{11}{5760}\left(\frac{z}{2}\right)^{10} + \frac{143}{806400}\left(\frac{z}{2}\right)^{12}\right\}-$$

Thus

$$\frac{J_0' J_0''}{z} = \frac{1}{4} - \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{5}{16}\left(\frac{z}{2}\right)^4 - \frac{7}{72}\left(\frac{z}{2}\right)^6 + \frac{7}{384}\left(\frac{z}{2}\right)^8 - \frac{11}{4800}\left(\frac{z}{2}\right)^{10} + \frac{143}{691200}\left(\frac{z}{2}\right)^{12} - \cdots\cdots$$

The particular integral for this term is

$$\frac{\phi_1}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{g}\right)^2 E\left\{\frac{1}{16}\left(\frac{z}{2}\right)^4 - \frac{1}{72}\left(\frac{z}{2}\right)^6 + \frac{5}{2304}\left(\frac{z}{2}\right)^8 - \frac{7}{28800}\left(\frac{z}{2}\right)^{10} + \frac{7}{345600}\left(\frac{z}{2}\right)^{12}\right.$$

$$\left. - \frac{11}{4\times 2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200\times 784\times 4}\left(\frac{z}{2}\right)^{16} - \cdots\cdots\right\}$$

The particular integral for the term $\frac{1}{4}\frac{gE}{R^2}J_0$ is

$$\frac{\phi_2}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{g}\right)\frac{1}{\beta^2}\, J_0$$

The particular integral for the term $-\frac{1}{4}\frac{gE}{R^2}\cos 2\theta\,(J_0 + 2J_0'')$ is

$$\frac{\phi_3}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{g}\right)\frac{1}{\beta^2}\cos 2\theta\; J_2$$

Then the total particular integral is

$$\frac{\phi_1+\phi_2+\phi_3}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4\frac{1}{\beta^4}\, gE\left[\left\{J_0 - \frac{g}{f}\left[\frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{18}\left(\frac{z}{2}\right)^6 + \frac{5}{576}\left(\frac{z}{2}\right)^8 - \frac{7}{7200}\left(\frac{z}{2}\right)^{10}\right.\right.\right.$$

$$\left. + \frac{7}{86400}\left(\frac{z}{2}\right)^{12} - \frac{11}{2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200\times 784}\left(\frac{z}{2}\right)^{16} - \cdots\cdots\right]\Bigg\}$$

$$\left. - J_2\cos 2\theta\right\} = \frac{\Phi}{R^2}$$

The stresses due to this particular integral are:

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z}\frac{\partial \Phi}{\partial z} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[\frac{J_0'}{z} - \frac{g}{64}\Big[\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6$$

$$- \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10} - \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big]$$

$$- \frac{J_2'}{z}\cos 2\theta\Big\}$$

$$\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[+\frac{J_2}{z^2}\,4\cos 2\theta\Big]$$

$$\widehat{rr}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[\frac{J_0'}{z} - \frac{g}{64}\Big\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6 - \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2}\right)\Big]$$

$$\widehat{\theta\theta}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[J_0'' - \frac{g}{64}\Big[\frac{3}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{12}\left(\frac{z}{2}\right)^4 + \frac{35}{288}\left(\frac{z}{2}\right)^6 - \frac{63}{2880}\left(\frac{z}{2}\right)^8 + \frac{77}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{143}{604800}\left(\frac{z}{2}\right)^{12} + \frac{2145}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big] - J_2''\cos 2\theta\Big\}$$

$$\widehat{\theta r}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[2\sin 2\theta\left(\frac{J_2'}{z} - \frac{J_2}{z^2}\right)\Big]$$

$$\frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2}\Big[\Big(\frac{J_0'}{z} - \nu J_0''\Big) - g\Big\{\frac{(1-3\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(1-5\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5(1-7\nu)}{288\times 4}\left(\frac{z}{2}\right)^6 - \frac{7(1-9\nu)}{2880\times 4}\left(\frac{z}{2}\right)^8 + \frac{7(1-11\nu)}{4\times 28800}\left(\frac{z}{2}\right)^{10} - \frac{11(1-13\nu)}{604800\times 4}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143(1-15\nu)}{4\times 135475200}\left(\frac{z}{2}\right)^{14} - - - - \Big\} - \cos 2\theta\Big\{\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2''\Big\}\Big]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_0}{\partial r}\right)^2\right\} = \frac{1}{2}\Big[\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{g}{\delta} J_0'\Big\{\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{g}{\delta} J_0' + 2\left(\frac{a}{R}\right)\sin^2\theta\Big\}\Big]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\Big[\Big(zJ_0' + g\frac{J_0'^2}{2}\Big) - zJ_0'\cos 2\theta\Big]$$

$$J_0'^2 = J_1^2 = \sum_n \frac{(-1)^n (2n+2)!\left(\frac{z}{2}\right)^{2n+2}}{n!\,(n+2)!\,(n+1)!\,(n+1)!}$$

$$= \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 + \frac{5}{12}\left(\frac{z}{2}\right)^6 - \frac{7}{72}\left(\frac{z}{2}\right)^8 + - - -$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\Big[\Big\{\Big(\frac{J_0'}{z} - zJ_0'\Big) - \nu J_0''\Big\} - g\Big\{\frac{3(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{5(5-\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5\times 7(7-\nu)}{1152}\left(\frac{z}{2}\right)^6 - \frac{7\times 9(9-\nu)}{11520}\left(\frac{z}{2}\right)^8 + \frac{7\times 11(11-\nu)}{115200}\left(\frac{z}{2}\right)^{10} - \frac{11\times 13(13-\nu)}{2419200}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143\times 15(15-\nu)}{541900800}\left(\frac{z}{2}\right)^{14} - - - \Big\} - \cos 2\theta\Big(\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2'' - zJ_0'\Big)\Big]$$

366

$$\frac{u_1}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{1}{\beta^3} g \left[\left\{ \left(\quad z J_0'' \quad \right) - \nu J_0' \right\} - g \left\{ \frac{(3-\nu)}{8}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{24}\left(\frac{z}{2}\right)^5 \right. \right.$$

$$+ \frac{5(7-\nu)}{576}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{5760}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{57600}\left(\frac{z}{2}\right)^{11} - \frac{11(13-\nu)}{1209600}\left(\frac{z}{2}\right)^{13}$$

$$\left. + \frac{143(15-\nu)}{270950400}\left(\frac{z}{2}\right)^{15} - \cdots \right\} - \cos 2\theta \left\{ -\nu J_2' - J_2' - z J_1' - J_0' \right\}$$

$$\int \left(\frac{J_2'}{z} - \frac{4J_2}{z^2} - z J_0' \right) dz = \int \left\{ \frac{J_2'}{z} - \left(J_2'' + \frac{J_2'}{z} + J_2 \right) - z J_0' \right\} dz$$

$$= -J_2' - \int \left(J_2 + z J_0' \right) dz$$

$$= -J_2' - \int \left\{ \frac{2J_1}{z} - J_0(z) + z J_0' \right\} dz = -J_2' + \int \left(\frac{J_0'}{z} - z J_0' \right) dz$$

$$+ \int \left(\frac{J_0'}{z} + J_0 \right) dz = -J_2' + z J_0'' + \int \left(\frac{J_0'}{z} + J_0 \right) dz = -J_2' + z J_0'' - J_0'$$

But

$$J_0 = J_2 - 2J_0''$$

$$\frac{J_0'}{z} = -J_2 + J_0''$$

$$J_0 + \frac{J_0'}{z} = -J_0''$$

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2} \left[\left\{ J_0'' - \nu \frac{J_0'}{z} \right\} - g \left\{ \frac{(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(5-\nu)}{48}\left(\frac{z}{2}\right)^4 + \cdots \right\} \right.$$

$$\left. + \cos 2\theta \left\{ \frac{J_2'}{z} + J_1' + \frac{J_0'}{z} + \nu \frac{J_2'}{z} \right\} \right]$$

$$-\frac{u}{\lambda}+\frac{1}{E}(\widehat{\theta\theta}_1-\widehat{rr}_1)=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''+\frac{\nu J_2'}{z}-\frac{4\nu J_2}{z^2}-\frac{J_2'}{z}-J_1'-\frac{J_0'}{z}-\frac{\nu J_2''}{z}\right\}\right] \qquad \underline{\underline{367}}$$

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''-\frac{J_2'}{z}-J_1'+\frac{J_1}{z}-\frac{4\nu J_2}{z^2}\right\}\right]$$

~~$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{J_2-\frac{4(1+\nu)J_2}{z^2}+\frac{J_1}{z}-J_1'\right\}\right]$$~~

$$\boxed{\frac{V_1}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{g}{\beta^3}\left[\frac{\sin 2\theta}{2}\left\{-zJ_2''-J_2'-zJ_1'+J_1-\frac{4\nu J_2}{z}\right\}\right]}$$

The total stress component can be expressed as

$$\widehat{rr}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{J_1}{z}-\frac{g}{4}\left\{\frac{1}{4}\left(\frac{z}{2}\right)^2-\frac{1}{12}\left(\frac{z}{2}\right)^4+\frac{5}{288}\left(\frac{z}{2}\right)^6-\frac{7}{2880}\left(\frac{z}{2}\right)^8+\frac{7}{28800}\left(\frac{z}{2}\right)^{10}\right.\right.$$
$$\left.\left.-\frac{11}{604800}\left(\frac{z}{2}\right)^{12}+\frac{143}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\right\}-\cos 2\theta\left\{2P_2+\frac{J_1}{z}-\frac{6J_2}{z^2}\right\}\right]$$

$$\widehat{\theta\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{1}{2}(J_0-J_2)-\frac{g}{4}\left\{\frac{3}{4}\left(\frac{z}{2}\right)^2-\frac{5}{12}\left(\frac{z}{2}\right)^4+\frac{35}{288}\left(\frac{z}{2}\right)^6-\frac{63}{2880}\left(\frac{z}{2}\right)^8\right.\right.$$
$$\left.+\frac{27}{28800}\left(\frac{z}{2}\right)^{10}-\frac{143}{604800}\left(\frac{z}{2}\right)^{12}+\frac{2145}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\right]+\cos 2\theta\left\{2P_2+12R_2z^2\right.$$
$$\left.\left.-\left(\frac{6}{z^2}-1\right)J_2+\frac{J_1}{z}\right\}\right]$$

$$\widehat{r\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\sin 2\theta\left\{2P_2+6R_2z^2-\frac{6J_2}{z^2}+\frac{3J_1}{z}\right\}\right]$$

The total deflection is

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{g}{\beta^3}\left[\frac{(1-\nu)}{2} Q_0 z + J_1 - zJ_0 + \nu J_1 - \frac{g}{4}\left\{\frac{(3-\nu)}{2}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{6}\left(\frac{z}{2}\right)^5\right.\right.$$

$$+ \frac{5(7-\nu)}{144}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{1440}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{14400}\left(\frac{z}{2}\right)^{11} - \frac{11(13-\nu)}{302400}\left(\frac{z}{2}\right)^{13}$$

$$+ \frac{143(15-\nu)}{67737600}\left(\frac{z}{2}\right)^{15} - \cdots\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2 z + 4\nu R_2 z^3 + J_1 - zJ_1'$$

$$\left. - (1+\nu) J_2'\Big\}\right]$$

$$\frac{v}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{g}{\beta^3}\left[\sin 2\theta\left\{2(1+\nu)P_2 z + 6(1+\nu)R_2 z^3 - \frac{zJ_2'}{2} - \frac{J_0'}{2} - \frac{zJ_1'}{2}\right.\right.$$

$$\left.\left. + \frac{1}{z}J_1 - \frac{2\nu J_2}{z}\right\}\right]$$

At $r = a$, $z = \beta$ $\qquad \frac{\beta}{2} = \frac{3.8317}{2} = 1.6159$

$$u_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\left[\frac{1}{2}Q_0 - \frac{g}{4}\left\{0.25 \times 1.6159^2 - 0.083333 \times 1.6159^4 + 0.00173611 \times 1.6159^6\right.\right.$$

$$- 0.0002430556 \times 1.6159^8 + 0.0000243056 \times 1.6159^{10} - 0.0000181878 \times 1.6159^{12}$$

$$\left.\left. + 0.0000010555 \times 1.6159^{14} - \cdots\right\} - \cos 2\theta\left\{2P_2 - 6\,\frac{0.4025}{3.8317^2}\right\}\right]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\left[\frac{1}{2}Q_0 - \frac{g}{4}\left\{0.25 \times 2.6111 - 0.083333 \times 6.8178 + 0.00173611 \times 17.8020\right.\right.$$

$$- 0.0002430556 \times 46.4828 + 0.0000243056 \times 121.37 - 0.0000181878 \times 316.91$$

$$\left.\left. + 0.0000010555 \times 827.48 - \cdots\right\} - \cos 2\theta\left\{2P_2 - 0.1645\right\}\right]$$

+0.65278		+1.95834	
−0.56815		−2.84075	
+0.03091		+0.21637	
−0.01130		−0.10170	
+0.00295		+0.03245	
−0.00576		−0.07488	
+0.00087		+0.01305	
	0.100		−0.824

$$\hat{M}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{1}{2}Q_0 - 0.02509 - \cos 2\theta\,(2P_2 - 0.1645)\right]$$

$$N_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{1}{2}Q_0 + 0.4027 + 0.2669 + \cos 2\theta\,\{2P_2 + 176.18\,R_2 + 0.2340\}\right]$$

$$\hat{N}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\,\{2P_2 + 88.09\,R_2 - 0.1645\}\right]$$

The non-uniform portion of $\frac{u}{R_a}$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[-\cos 2\theta\,\{2(1+\nu)P_2 + 58.7276\,\nu\,R_2 + 0.4027 + (1+\nu)\,0.05483\}\right]$$

$$\frac{v}{R_a} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\,\{2(1+\nu)P_2 + 88.0914(1+\nu)R_2 + 0.3478 - 0.054838\,\nu\}\right]$$

Put $\quad \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2} = \eta$

Then the stress + displacement conditions give

$$\frac{1}{2}+r_0 = \eta g\left\{\frac{1}{2}Q_0 - 0.0250g\right\} \qquad (1)$$

$$\frac{1}{2}-r_0 = \eta g\left\{\frac{1}{2}Q_0 + 0.4027 + 0.206g\right\} \qquad (2)$$

$$(3)\quad \frac{1}{2} - 6q_{12} - 4S_2 = \eta g\left\{0.1645 - 2P_2\right\}$$

$$(4)\quad 6q_{12} - \frac{1}{2} = \eta g\left\{2P_2 + 176.18P_2 + 0.2380\right\}$$

$$(5)\quad \frac{1}{2} + 6q_{12} + 2S_2 = \eta g\left\{0.1145 - 9P_2 - 88.01P_2\right\}$$

$$(6)\quad \frac{1}{2}(1+\lambda) + 2(1+\lambda)q_{12} + 4S_2 = \eta g\left\{-2(1+\lambda)P_2 - 58.7226\lambda P_2 - 0.4022 - (1+\lambda)0.0548 \right\}$$

$$(7)\quad 2(1+\lambda)q_{12} - \frac{1}{2}(1+\lambda) = \eta g\left\{2(1+\lambda)P_2 + [illegible]\,P_2 + 0.5[illegible] - 0.05[illegible]\right\}$$

---

from (1) + (2)

$$1 = \eta g\left\{Q_0 + 0.4027 + 0.181g\right\}$$

$$\boxed{Q_0 = \frac{1}{\eta g} - (0.4027 + 0.181g)}$$

$$r_0 = \eta g\left\{-(0.20135 + 0.0905g) - 0.[illegible]g\right\} = -\eta g\left\{0.20135 + 0.1155g\right\}$$

$$\boxed{r_0 = -\eta g(0.2014 + 0.1155g)}$$

$$q_2 + 0.6667\,S_2 - 0.08333 = \eta g \{ 0.3333\,P_2 - 0.02742 \}$$

$$q_2 + \quad 0 - 0.08333 = \eta g \{ 0.3333\,P_2 + 29.36\,R_2 + 0.03967 \}$$

$$q_2 + 0.3333\,S_2 + 0.08333 = \eta g \{ -0.3333\,P_2 - 14.68\,R_2 + 0.02742 \}$$

$$q_2 + 1.5385\,S_2 + 0.2500 = \eta g \{ - P_2 - 6.7760\,R_2 - 0.1823 \}$$

$$q_2 + \quad 0 \quad - 0.2500 = \eta g \{ P_2 + 44.0457\,R_2 + 0.1275 \}$$

$$0.6667\,S_2 + \quad 0 \quad = \eta g \{ - 29.36\,R_2 - 0.06709 \}$$

$$0.3333\,S_2 + 0.1667 = \eta g \{ -0.6667\,P_2 - 44.04\,R_2 - 0.01225 \}$$

$$1.2052\,S_2 + 0.1667 = \eta g \{ -0.6667\,P_2 + 7.904\,R_2 - 0.2097 \}$$

$$1.5385\,S_2 + 0.5000 = \eta g \{ -2\,P_2 - 50.83\,R_2 - 0.3098 \}$$

$$S_2 + \quad 0 = \eta g \{ -44.04\,R_2 - 0.1006 \}$$

$$S_2 + 0.5000 = \eta g \{ -2\,P_2 - 132.12\,R_2 - 0.03675 \}$$

$$S_2 + 0.1383 = \eta g \{ -0.5532\,P_2 + 6.558\,R_2 - 0.1740 \}$$

$$S_2 + 0.3250 = \eta g \{ -1.3\,P_2 - 33.038\,R_2 - 0.2014 \}$$

$$0.5000 = \eta g \{ -2\,P_2 - 88.08\,R_2 + 0.06365 \}$$

$$0.3617 = \eta g \{ -1.4468\,P_2 - 138.68\,R_2 + 0.1372 \}$$

$$0.1867 = \eta g \{ -0.7466\,P_2 - 37.596\,R_2 - 0.0274 \}$$

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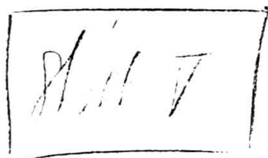

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For the circular region,

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta \right\}$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta - f_1\left[1 - \left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1 - \left(\frac{r}{a}\right)^3\right]^2 - f_3\left[1 - \left(\frac{r}{a}\right)^4\right]^2 \right\}$$

where f_1, f_2, f_3, are amplitudes.

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta + 4f_1\left[1 - \left(\frac{r}{a}\right)^2\right] + 6f_2\left[1 - \left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[1 - \left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)_0 = \frac{1}{2}\frac{1}{R}\left\{ -2 \sin^2\theta \right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta + 4f_1\left[1 - 3\left(\frac{r}{a}\right)^2\right] + 6f_2\left[2 - 5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[3 - 7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta \right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{ -\cos 2\theta \right\}$$

$$\frac{32^2}{14} = \frac{32 \times 16}{7}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left\{\sin^2\theta - 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] - 3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) - 4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}\left\{\sin^2\theta - 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] - 3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) - 4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$= \frac{1}{R^2}\left\{2f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 3f_2\left[1.5-3\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[2-4\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}\underline{(1-\cos 2\theta)}$$

$$-\frac{1}{R^2}\left\{4f_1^2\left[1-4\left(\frac{r}{a}\right)^2+3\left(\frac{r}{a}\right)^4\right] + 6f_1f_2\left[1-3\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^3+3\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right) + 8f_1f_3\left[1-3\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^4+3\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2\right.$$

$$+ 6f_1f_2\left[2-2\left(\frac{r}{a}\right)^2-5\left(\frac{r}{a}\right)^3+5\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right) + 8f_1f_3\left[3-3\left(\frac{r}{a}\right)^2-7\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2$$

$$+ 9f_2^2\left[2-7\left(\frac{r}{a}\right)^3+5\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2 + 12f_2f_3\left[2-5\left(\frac{r}{a}\right)^3-2\left(\frac{r}{a}\right)^4+5\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3$$

$$\left.+12f_2f_3\left[3-3\left(\frac{r}{a}\right)^3-7\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3 + 16f_3^2\left[3-10\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^8\right]\left(\frac{r}{a}\right)^4\right\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$= \frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

The terms for the particular integral is then, <u>multiplied by</u> $\underline{R^2}$,

$$2f_1\left[1-2\left(\frac{r}{a}\right)^2\right]+3f_2\left[1.5-3\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)+4f_3\left[2-4\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2$$

$$-4f_1^2\left[1-4\left(\frac{r}{a}\right)^2+3\left(\frac{r}{a}\right)^4\right]-6f_1f_2\left[3-5\left(\frac{r}{a}\right)^2-6\left(\frac{r}{a}\right)^3+8\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right)-8f_1f_3\left[4-6\left(\frac{r}{a}\right)^2-8\left(\frac{r}{a}\right)^4+10\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2$$

$$-9f_2^2\left[2-7\left(\frac{r}{a}\right)^3+5\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2-12f_2f_3\left[5-8\left(\frac{r}{a}\right)^3-9\left(\frac{r}{a}\right)^4+12\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3-16f_3^2\left[3-10\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^8\right]\left(\frac{r}{a}\right)^4$$

$$-\cos 2\theta\left\{2f_1\left(\frac{r}{a}\right)^2-3f_2\left[0.5-2\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)-4f_3\left[1-3\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$=2f_1(1-2f_1)+(4.5f_2-18f_1f_2)\left(\frac{r}{a}\right)+(-4f_1+8f_3+16f_1^2-32f_1f_3-18f_2^2)\left(\frac{r}{a}\right)^2$$

$$+(30f_1f_2-60f_2f_3)\left(\frac{r}{a}\right)^3+(-9f_2-12f_1^2+36f_1f_2+48f_1f_3-48f_3^2)\left(\frac{r}{a}\right)^4$$

$$+(63f_2^2)\left(\frac{r}{a}\right)^5+(-16f_3-48f_1f_2+64f_1f_3+96f_2f_3)\left(\frac{r}{a}\right)^6+(108f_2f_3)\left(\frac{r}{a}\right)^7+(-80f_1f_3-45f_2^2+160f_3^2)\left(\frac{r}{a}\right)^8$$

$$+(-144f_2f_3)\left(\frac{r}{a}\right)^{10}+-112f_3^2\left(\frac{r}{a}\right)^{12}$$

$$+\cos 2\theta\left\{1.5f_2\left(\frac{r}{a}\right)+(4f_3-2f_1)\left(\frac{r}{a}\right)^2-6f_2\left(\frac{r}{a}\right)^4-12f_3\left(\frac{r}{a}\right)^6\right\}$$

$$\frac{\Phi}{r^2} = E\left(\frac{a}{R}\right)^4 \Big[\frac{1}{4} q_0 \left(\frac{r}{a}\right)^2 + \frac{f_1}{32}(1-2f_1)\left(\frac{r}{a}\right)^4 + 0.02\, f_2 (1-4f_1)\left(\frac{r}{a}\right)^5 + \frac{1}{144}\left(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{6}{245} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^7 + \frac{1}{768}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{63} f_2^2 \left(\frac{r}{a}\right)^9$$

$$+ \frac{1}{400}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^{10} + \frac{12}{1089} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \frac{1}{2880}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{12}$$

$$- \frac{1}{196} f_2 f_3 \left(\frac{r}{a}\right)^{14} - \frac{1}{448} f_3^2 \left(\frac{r}{a}\right)^{16} + \cos 2\theta \Big\{ \frac{1}{70} f_2 \left(\frac{r}{a}\right)^5 + \frac{1}{192}(2f_3 - f_1)\left(\frac{r}{a}\right)^6 - \frac{1}{320} f_2 \left(\frac{r}{a}\right)^8$$

$$- \frac{1}{480} f_3 \left(\frac{r}{a}\right)^{10} + p_2 \left(\frac{r}{a}\right)^2 + r_2 \left(\frac{r}{a}\right)^4 \Big\} \Big]$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}\left(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{6}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{1}{40}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^8 + \frac{12}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{1}{240}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{1}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{14} f_2 \left(\frac{r}{a}\right)^3 + \frac{1}{32}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{40} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{1}{48} f_3 \left(\frac{r}{a}\right)^8 + 2 p_2 + 4 r_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$
\begin{aligned}
\widehat{rr} = E\left(\frac{a}{R}\right)^2 \Big[& \frac{1}{2}q_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)\left(\frac{r}{a}\right)^4 \\
& + \frac{6}{35}f_2(f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}(12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2)\left(\frac{r}{a}\right)^6 + \frac{1}{7}f_2^2\left(\frac{r}{a}\right)^7 \\
& + \frac{1}{40}(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2)\left(\frac{r}{a}\right)^8 + \frac{12}{99}f_2f_3\left(\frac{r}{a}\right)^9 + \frac{1}{240}(32f_3^2 - 16f_1f_3 - 9f_2^2)\left(\frac{r}{a}\right)^{10} \\
& - \frac{1}{14}f_2f_3\left(\frac{r}{a}\right)^{12} - \frac{1}{28}f_3^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{70}f_2\left(\frac{r}{a}\right)^3 + \frac{1}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{80}f_2\left(\frac{r}{a}\right)^6 - \frac{1}{80}f_3\left(\frac{r}{a}\right)^8 \\
& - 2p_2 \Big\} \Big]
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Big[& \frac{1}{2}q_0 + \frac{3}{8}f_1(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{4}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{5}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)\left(\frac{r}{a}\right)^4 \\
& + \frac{36}{35}f_2(f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{7}{96}(12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2)\left(\frac{r}{a}\right)^6 + \frac{8}{7}f_2^2\left(\frac{r}{a}\right)^7 \\
& + \frac{9}{40}(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2)\left(\frac{r}{a}\right)^8 + \frac{120}{99}f_2f_3\left(\frac{r}{a}\right)^9 + \frac{11}{240}(32f_3^2 - 16f_1f_3 - 9f_2^2)\left(\frac{r}{a}\right)^{10} \\
& - \frac{13}{14}f_2f_3\left(\frac{r}{a}\right)^{12} - \frac{15}{28}f_3^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{20}{70}f_2\left(\frac{r}{a}\right)^3 + \frac{15}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{14}{80}f_2\left(\frac{r}{a}\right)^6 \\
& - \frac{15}{80}f_3\left(\frac{r}{a}\right)^8 + 2p_2 + 12n_2\left(\frac{r}{a}\right)^2 \Big\} \Big]
\end{aligned}
$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2 \left[\sin 2\theta \left\{ \frac{\ell}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{5}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{3.5}{80} f_2\left(\frac{r}{a}\right)^6 - \frac{3}{80} f_3 \left(\frac{r}{a}\right)^8 + 2p_2 + 6 r_2 \left(\frac{r}{a}\right)^2 \right\}\right]$$

$$\widehat{rr} - \nu\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \left[\frac{1}{2}(1-\nu)q_0 + \frac{(1-3\nu)}{8} f_1 (1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{(1-4\nu)}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{(1-5\nu)}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)\left(\frac{r}{a}\right)^4 \right.$$

$$+ \frac{6(1-6\nu)}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{(1-7\nu)}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2)\left(\frac{r}{a}\right)^6 + \frac{(1-8\nu)}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{(1-9\nu)}{40}(4 f_1 f_3 + 6 f_1 f_3 - f_3 - 3 f_1 f_2)\left(\frac{r}{a}\right)^8 + \frac{12(1-10\nu)}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{(1-11\nu)}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{(1-13\nu)}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{(1-15\nu)}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \left\{ \frac{(1-20\nu)}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{(1-15\nu)}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{(1-14\nu)}{80} f_2 \left(\frac{r}{a}\right)^6 \right.$$

$$\left.\left. - \frac{(1-15\nu)}{80} f_3 \left(\frac{r}{a}\right)^8 - 2(1+\nu)p_2 - 12\nu r_2 \left(\frac{r}{a}\right)^2 \right\}\right]$$

347

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\frac{1}{2}\left(\frac{r}{R}\right)^2\Bigg[(\cos 2\theta-1)\Big\{2f_1\left[1-\left(\frac{r}{a}\right)^2\right]+3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)+4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\Big\}$$

$$+\Big\{4f_1^2\left[1-2\left(\frac{r}{a}\right)^2+\left(\frac{r}{a}\right)^4\right]+12f_1f_2\left[1-\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^3+\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right)+16f_1f_3\left[1-\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2$$

$$+9f_2^2\left[1-2\left(\frac{r}{a}\right)^3+\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2+24f_2f_3\left[1-\left(\frac{r}{a}\right)^3-\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3+16f_3^2\left[1-2\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^8\right]\left(\frac{r}{a}\right)^4\Big\}\Bigg]$$

$$=\frac{1}{2}\left(\frac{a}{R}\right)^2\Bigg[\cos 2\theta\Big\{2f_1\left(\frac{r}{a}\right)^2+3f_2\left(\frac{r}{a}\right)^3+2(2f_3-f_1)\left(\frac{r}{a}\right)^4-3f_2\left(\frac{r}{a}\right)^6-4f_3\left(\frac{r}{a}\right)^8\Big\}$$

$$+2f_1(2f_1-1)\left(\frac{r}{a}\right)^2+(12f_1f_2-3f_2)\left(\frac{r}{a}\right)^3+(-8f_1^2+16f_1f_3+9f_2^2-4f_3+2f_1)\left(\frac{r}{a}\right)^4$$

$$+(-12f_1f_2+24f_2f_3)\left(\frac{r}{a}\right)^5+(4f_1^2-12f_1f_2-16f_1f_3+16f_3^2+3f_2)\left(\frac{r}{a}\right)^6$$

$$-18f_2^2\left(\frac{r}{a}\right)^7+(12f_1f_2-16f_1f_3-4f_3-24f_2f_3)\left(\frac{r}{a}\right)^8+(-24f_2f_3)\left(\frac{r}{a}\right)^9+(16f_1f_3+9f_2^2-32f_3^2)\left(\frac{r}{a}\right)^{10}$$

$$+24f_2f_3\left(\frac{r}{a}\right)^{12}+16f_3^2\left(\frac{r}{a}\right)^{14}\Big\}\Bigg]$$

$$\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\left(\frac{a}{R}\right)^2\Big[\cos 2\theta\Big\{\; f_1\left(\frac{r}{a}\right)^2+\frac{3}{2}f_2\left(\frac{r}{a}\right)^3+(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3}{2}f_2\left(\frac{r}{a}\right)^6-2f_3\left(\frac{r}{a}\right)^8\Big\}$$

$$-f_1(1-2f_1)\left(\frac{r}{a}\right)^2-\frac{3}{2}f_2(1-4f_2)\left(\frac{r}{a}\right)^3-\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$-6f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5+\frac{1}{2}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6-9f_2^2\left(\frac{r}{a}\right)^7$$

$$-2\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8-12f_2f_3\left(\frac{r}{a}\right)^9-\frac{1}{2}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+12f_2f_3\left(\frac{r}{a}\right)^{12}+8f_3^2\left(\frac{r}{a}\right)^{14}\Big\}\Big]$$

$$\frac{\partial u}{\partial r}=\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)q_0+\frac{3(3-\nu)}{8}f_1(1-2f_1)\left(\frac{r}{a}\right)^2+\frac{4(4-\nu)}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3$$

$$+\frac{5(5-\nu)}{24}\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4+\frac{36(6-\nu)}{35}f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5$$

$$+\frac{7(7-\nu)}{96}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6+\frac{8(8-\nu)}{7}f_2^2\left(\frac{r}{a}\right)^7$$

$$+\frac{9(9-\nu)}{40}\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8+\frac{120(10-\nu)}{99}f_2f_3\left(\frac{r}{a}\right)^9+\frac{11(11-\nu)}{240}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$-\frac{13(13-\nu)}{14}f_2f_3\left(\frac{r}{a}\right)^{12}-\frac{15(15-\nu)}{28}f_3^2\left(\frac{r}{a}\right)^{14}$$

$$- \cos 2\theta \left\{ f_1 \left(\frac{r}{a}\right)^2 + \frac{(104+90\nu)}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{(95+15\nu)}{96} (2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{80} f_2 \left(\frac{r}{a}\right)^6 - \frac{(159+15\nu)}{80} f_3 \left(\frac{r}{a}\right)^8 \right.$$

$$\left. + 2(1+\nu) p_2 + 12\nu r_2 \left(\frac{r}{a}\right)^2 \right\} \Bigg]$$

$$\frac{u}{R} = \left(\frac{a}{R}\right)^3 \Bigg[\frac{1}{2}(1-\nu) q \left(\frac{r}{a}\right) + \frac{(3-\nu)}{8} f_1 (1-2f_1)\left(\frac{r}{a}\right)^3 + \frac{(4-\nu)}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2)\left(\frac{r}{a}\right)^5$$

$$+ \frac{6(6-\nu)}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^6 + \frac{(7-\nu)}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2)\left(\frac{r}{a}\right)^7 + \frac{(8-\nu)}{7} f_2^2 \left(\frac{r}{a}\right)^8$$

$$+ \frac{(9-\nu)}{40}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)\left(\frac{r}{a}\right)^9 + \frac{12(10-\nu)}{99} f_2 f_3 \left(\frac{r}{a}\right)^{10} + \frac{(11-\nu)}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2)\left(\frac{r}{a}\right)^{11}$$

$$- \frac{(13-\nu)}{14} f_2 f_3 \left(\frac{r}{a}\right)^{13} - \frac{(15-\nu)}{28} f_3^2 \left(\frac{r}{a}\right)^{15}$$

$$- \cos 2\theta \left\{ \frac{1}{3} f_1 \left(\frac{r}{a}\right)^3 + \frac{(26+5\nu)}{70} f_2 \left(\frac{r}{a}\right)^4 + \frac{(19+3\nu)}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{80} f_2 \left(\frac{r}{a}\right)^7 - \frac{(53+5\nu)}{240} f_3 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + 2(1+\nu) p_2 \left(\frac{r}{a}\right) + 4\nu r_2 \left(\frac{r}{a}\right)^3 \right\} \Bigg]$$

380

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \Big[\cos 2\theta \Big\{ \frac{1}{3} f_1 \left(\frac{r}{a}\right)^2 + \frac{(23+2\nu)}{35} f_2 \left(\frac{r}{a}\right)^3 + \frac{(17+\nu)}{48} (2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{(31+\nu)}{80} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{(49+\nu)}{120} f_3 \left(\frac{r}{a}\right)^8 + 4(1+\nu) p_2 + 4(3+\nu) r_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \Big[\sin 2\theta \Big\{ \frac{1}{6} f_1 \left(\frac{r}{a}\right)^3 + \frac{(23+2\nu)}{70} f_2 \left(\frac{r}{a}\right)^4 + \frac{(17+\nu)}{96} (2f_3 - f_1)\left(\frac{r}{a}\right)^5 - \frac{(31+\nu)}{160} f_2 \left(\frac{r}{a}\right)^7$$

$$- \frac{(49+\nu)}{240} f_3 \left(\frac{r}{a}\right)^8 + 2(1+\nu) p_2 \left(\frac{r}{a}\right) + 2(3+\nu) r_2 \left(\frac{r}{a}\right)^3 \Big\} \Big]$$

Thus the non-uniform portions are (i.e. terms independent of θ, at $r = a$)

$$\hat{r}_a = \cos 2\theta \left\{ \frac{f_2}{560} - \frac{f_1}{96} + \frac{f_3}{120} - 2p_2 \right\}$$

$$\hat{\theta}_a = \cos 2\theta \left\{ \frac{31}{280} f_2 - \frac{5}{32} f_1 + \frac{1}{8} f_3 + 2p_2 + 12 r_2 \right\}$$

$$\hat{r\theta}_a = \sin 2\theta \left\{ \frac{39.5}{560} f_2 - \frac{5}{96} f_1 + \frac{1}{15} f_3 + 2p_2 + 6 r_2 \right\}$$

381

$$\left(\frac{u}{R}\right)_a = -\cos 2\theta \left\{ \frac{(13-3\nu)}{96} f_1 + \frac{(89+26\nu)}{560} f_2 + \frac{(21+5\nu)}{120} f_3 + 2(1+\nu) p_2 + 4\nu r_2 \right\}$$

25

$$\left(\frac{v}{R}\right)_a = \sin 2\theta \left\{ -\frac{(1+\nu)}{96} f_1 + \frac{(151+15\nu)}{1120} f_2 + \frac{(9+\nu)}{60} f_3 + 2(1+\nu) p_2 + 2(3+\nu) r_2 \right\}$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ \frac{f_2}{560} - \frac{f_1}{96} + \frac{f_3}{120} - 2p_2 \right\}$$

$$6q_2 - \frac{1}{2} = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ \frac{31}{280} f_2 - \frac{5}{32} f_1 + \frac{1}{8} f_3 + 2p_2 + 12 r_2 \right\}$$

$$\frac{1}{2} + 6q_2 + 2s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ -\frac{79}{1120} f_2 + \frac{5}{96} f_1 - \frac{1}{15} f_3 - 2p_2 - 6r_2 \right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu) q_2 + 4s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{(89+26\nu)}{560} f_2 - \frac{(21+5\nu)}{120} f_3 - 2(1+\nu) p_2 - 4\nu r_2 \right\}$$

25

$$2(1+\nu) q_2 - \frac{1}{2}(1+\nu) = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2 \left\{ -\frac{(1+\nu)}{96} f_1 + \frac{(151+15\nu)}{1120} f_2 + \frac{(9+\nu)}{60} f_3 + 2(1+\nu) p_2 + 2(3+\nu) r_2 \right\}$$

Put $\frac{E}{\sigma}\left(\frac{a}{R}\right)^2 = \eta$

3f2

$$1)\ q_2 + 0.666667 S_2 - 0.083333 = \eta\{0.333333 p_2 + 0 + 0.00173611 f_1 - 0.00029762 f_2 - 0.00138889 f_3\}$$

$$2)\ q_2 + 0 - 0.083333 = \eta\{0.333333 p_2 + 2 r_2 - 0.02604167 f_1 + 0.01845238 f_2 + 0.02083333 f_3\}$$

$$3)\ q_2 + 0.333333 S_2 + 0.083333 = \eta\{-0.333333 p_2 - r_2 + 0.00868056 f_1 - 0.01175595 f_2 - 0.01111111 f_3\}$$

$$4)\ q_2 + 1.5384615 S_2 + 0.25000 = \eta\{- p_2 - 0.4615382 r_2 - 0.04847756 f_1 - 0.06648352 f_2 - 0.07211538 f_3\}$$

$$5)\ q_2 + 0 - 0.250000 = \eta\{p_2 + 2.5384612 r_2 - 0.00520833 f_1 + 0.05442995 f_2 + 0.05961538 f_3\}$$

$$0.666667 S_2 + 0 = \eta\{0 - 2 r_2 + 0.02777778 f_1 - 0.01875000 f_2 - 0.02222222 f_3\}$$

$$0.333333 S_2 + 0.166667 = \eta\{-0.666667 p_2 - 3 r_2 + 0.03472222 f_1 - 0.03020833 f_2 - 0.03194444 f_3\}$$

$$1.2051282 S_2 + 0.166667 = \eta\{-0.666667 p_2 + 1.538462 r_2 - 0.05715812 f_1 - 0.05472757 f_2 - 0.06100427 f_3\}$$

$$1.5384615 S_2 + 0.5000 = \eta\{-2 p_2 - 3 r_2 - 0.04326923 f_1 - 0.12091347 f_2 - 0.13173076 f_3\}$$

$$6)\ S_2 + 0 = \eta\{0 - 3 r_2 + 0.04166667 f_1 - 0.02812500 f_2 - 0.03333333 f_3\}$$

$$7)\ S_2 + 0.50000 = \eta\{-2 p_2 - 9 r_2 + 0.10416667 f_1 - 0.09062500 f_2 - 0.09583333 f_3\}$$

$$8)\ S_2 + 0.1382978 = \eta\{-0.5531915 p_2 + 0.4468091 r_2 - 0.047429079 f_1 - 0.04541224 f_2 - 0.05062056 f_3\}$$

$$9)\ S_2 + 0.32500 = \eta\{-1.3 p_2 - 1.95 r_2 - 0.02812500 f_1 - 0.07859376 f_2 - 0.08562500 f_3\}$$

$$0.500000 = \eta\left\{-2p_2 - 6r_2 + 0.06250000f_1 - 0.06250000f_2 - 0.06250000f_3\right\}$$

$$0.36170222 = \eta\left\{-1.4468085p_2 - 9.446809r_2 + 0.15159575f_1 - 0.04521276f_2 - 0.04521277f_3\right\}$$

$$0.18670222 = \eta\left\{-0.7468085p_2 - 2.396809r_2 + 0.01930408f_1 - 0.03318152f_2 - 0.03500444f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 3r_2 + 0.031250000f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 6.529412r_2 + 0.10477947f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 3.209403r_2 + 0.02584877f_1 - 0.04443110f_2 - 0.04687205f_3\right\}$$

$$3.529412\, r_2 = 0.07352942f_1 - 0\, f_2 + 0\, f_3$$

$$3.320009\, r_2 = 0.07893065f_1 + 0.01318110f_2 + 0.01562205f_3$$

$$\boxed{r_2 = 0.02083333f_1 - 0} + 0\, f_3$$

$$r_2 = 0.02377423f_1 + 0.00397020f_2 + 0.00470542f_3$$

$$0 = 0.00294090f_1 + 0.00397020f_2 + 0.00470542f_3$$

$$\boxed{-f_3 = 0.625000f_1 + 0.843750f_2}$$

$$2\eta p_2 = -0.500000 + \eta \begin{Bmatrix} f_1 & f_2 \\ -0.1985294 & \\ +0.1360294 & -0.0625000 \\ +0.0390625 & +0.0527344 \end{Bmatrix}$$

$$\boxed{\eta p_2 = -0.250000 - \eta \{ 0.0117188 f_1 + 0.0048828 f_2 \}}$$

$$3S_2 = \eta \begin{Bmatrix} f_1 & f_2 \\ +0.0299303 & +0.0124667 \\ -0.2406915 & \\ +0.0984043 & -0.1641622 \\ +0.1123670 & +0.1516955 \end{Bmatrix}$$

$$\boxed{S_2 = -\eta \{ 0 + 0 \}}$$

$$4q_2 = \eta \begin{Bmatrix} f_1 & f_2 \\ +0.0378125 & +0.0032552 \\ +0.0112179 & \\ -0.0641026 & -0.0600648 \\ +0.0398638 & +0.0538161 \end{Bmatrix}$$

$$\boxed{q_2 = -\eta \{ 0.0013021 f_1 + 0.0007534 f_2 \}}$$

CHECK

f_1	f_2
-0.0117188	-0.0048880
$+0.0522846$	$-\ 0$
-0.0052083	$+0.0544299$
-0.0372596	-0.0503004

$r_2 = 0.02083333 f_1$ — O.K.

$$\frac{1}{2} + \nu_0 = \eta\Big\{\frac{1}{2}q_0 + \frac{f_1}{8}(1-2f_1) + \frac{1}{10}f_2(1-4f_1) + \frac{1}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)$$

$$+ \frac{6}{35}f_2(f_1 - 2f_3) + \frac{1}{96}(12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{7}f_2^2$$

$$+ \frac{1}{40}(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2) + \frac{12}{77}f_2f_3 + \frac{1}{240}(32f_3^2 - 16f_1f_3 - 9f_2^2) - \frac{1}{14}f_2f_3 - \frac{1}{24}f_3^2$$

$$\frac{1}{2} - \nu_0 = \eta\Big\{\frac{1}{2}q_0 + \frac{3}{8}f_1(1-2f_1) + \cdots\Big\}$$

$$\therefore \quad q_0 = \frac{1}{\eta} - \frac{1}{2}f_1(1-2f_1) - \frac{1}{2}f_2(1-4f_1) - \frac{1}{4}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)$$

$$- \frac{6}{5}f_2(f_1 - 2f_3) - \frac{1}{12}(12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2) - \frac{9}{7}f_2^2$$

$$- \frac{1}{4}(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2) - \frac{4}{3}f_2f_3 - \frac{1}{20}(32f_3^2 - 16f_1f_3 - 9f_2^2) + f_2f_3 + \frac{4}{7}f_3^2$$

$$\nu_0 = -\eta\Big\{\frac{1}{8}f_1(1-2f_1) + \frac{3}{20}f_2(1-4f_1) + \frac{1}{12}(2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2)$$

$$+ \frac{3}{7}f_2(f_1 - 2f_3) + \frac{1}{32}(12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{2}f_2^2 + \frac{1}{10}(4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2)$$

$$+ \frac{6}{11}f_2f_3 + \frac{1}{48}(32f_3^2 - 16f_1f_3 - 9f_2^2) - \frac{3}{7}f_2f_3 - \frac{1}{4}f_3^2\Big\}$$

386

Putting $f_1(1-2f_1)=A$, $f_2(1-4f_1)=B$, $(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2)=C$,

$f_2(f_1-2f_3)=D$, $(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2)=E$, $f_2^2=F$,

$(4f_1f_3+6f_2f_3-f_3-3f_1f_2)=G$, $f_2f_3=H$, $(32f_3^2-16f_1f_3-9f_2^2)=I$,

$f_2f_3=J$, $f_3^2=K$

$$\widehat{rr}+\widehat{\theta\theta}=E\left(\frac{a}{R}\right)^2\Big[\Big\{\frac{1}{7}-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{4}C-\frac{6}{5}D-\frac{1}{12}E-\frac{9}{7}F-\frac{1}{4}G-\frac{4}{3}H-\frac{1}{20}I$$

$$q_0$$

$$+J+\frac{4}{7}K\Big\}+\frac{1}{2}A\left(\frac{r}{a}\right)^2+\frac{1}{2}B\left(\frac{r}{a}\right)^3+\frac{1}{4}C\left(\frac{r}{a}\right)^4+\frac{6}{5}D\left(\frac{r}{a}\right)^5+\frac{1}{12}E\left(\frac{r}{a}\right)^6+\frac{9}{7}F\left(\frac{r}{a}\right)^7$$

$$+\frac{1}{4}G\left(\frac{r}{a}\right)^8+\frac{4}{3}H\left(\frac{r}{a}\right)^9+\frac{1}{20}I\left(\frac{r}{a}\right)^{10}-J\left(\frac{r}{a}\right)^{12}-\frac{4}{7}K\left(\frac{r}{a}\right)^{14}+$$

$$\cos 2\theta\Big\{\frac{3}{10}f_2\left(\frac{r}{a}\right)^3+\frac{1}{6}(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3}{16}f_2\left(\frac{r}{a}\right)^6-\frac{1}{5}f_3\left(\frac{r}{a}\right)^8+0\quad+12r_2\left(\frac{1}{a}\right)^2\Big\}\Big]$$

$$\int_0^{2\pi}(\widehat{rr}+\widehat{\theta\theta})^2\,d\theta=\pi E^2\left(\frac{a}{R}\right)^4\Big[2\Big\{q_0^2+Aq_0\left(\frac{r}{a}\right)^2+Bq_0\left(\frac{r}{a}\right)^3+\left(\frac{A^2}{4}+\frac{1}{2}Cq_0\right)\left(\frac{r}{a}\right)^4+\left(\frac{1}{2}AB+\frac{12}{5}Dq_0\right)\left(\frac{r}{a}\right)^5$$

$$+\left(\frac{1}{4}B^2+\frac{1}{6}Eq_0\right)\left(\frac{r}{a}\right)^6+\left(\frac{1}{4}BC+\frac{18}{7}Fq_0\right)\left(\frac{r}{a}\right)^7+\left(\frac{1}{16}C^2+\frac{1}{2}Gq_0\right)\left(\frac{r}{a}\right)^8+($$

$$\int_0^{2\pi} (\overline{AA} + \overline{OO})^2 \, d\ell = \pi E^2 \left(\frac{a}{R}\right)^4 \Big[2 \Big\{ q_0^2 + A q_0 \left(\frac{\lambda}{a}\right)^2 + B q_0 \left(\frac{\lambda}{a}\right)^3 + \left(\frac{A^2}{4} + \frac{1}{2} C q_0 \right) \left(\frac{\lambda}{a}\right)^4 + \left(\frac{1}{2} AB + \frac{12}{5} D q_0 \right) \left(\frac{\lambda}{a}\right)^5$$

$$+ \left(\frac{1}{4} B^2 + \frac{1}{4} AC + \frac{1}{6} E q_0 \right) \left(\frac{\lambda}{a}\right)^6 + \left(\frac{1}{4} BC + \frac{6}{5} AD + \frac{18}{7} F q_0 \right) \left(\frac{\lambda}{a}\right)^7 + \left(\frac{1}{16} C^2 + \frac{6}{5} BD + \frac{1}{12} AE + \frac{1}{2} G q_0 \right) \left(\frac{\lambda}{a}\right)^8$$

$$+ \left(\frac{3}{5} CD + \frac{1}{12} BE + \frac{9}{7} AF + \frac{8}{3} H q_0 \right) \left(\frac{\lambda}{a}\right)^9 + \left(\frac{36}{25} D^2 + \frac{1}{24} CE + \frac{9}{7} BF + \frac{1}{4} AG + \frac{1}{10} I q_0 \right) \left(\frac{\lambda}{a}\right)^{10}$$

$$+ \left(\frac{1}{5} DE + \frac{9}{14} CF + \frac{1}{4} BG + \frac{4}{3} AH \right) \left(\frac{\lambda}{a}\right)^{11} +$$

$$+ \left(\frac{1}{144} E^2 + \frac{108}{35} DF + \frac{1}{8} CG + \frac{4}{3} BH + \frac{1}{10} AI - 2 J q_0 \right) \left(\frac{\lambda}{a}\right)^{12} + \left(\frac{3}{14} EF + \frac{3}{5} DG + \frac{2}{3} CH \right.$$

$$\left. + \frac{1}{20} BI \right) \left(\frac{\lambda}{a}\right)^{13} + \left(\frac{81}{49} F^2 + \frac{1}{24} EG + \frac{48}{15} DH + \frac{1}{40} CI - AJ - \frac{8}{7} K q_0 \right) \left(\frac{\lambda}{a}\right)^{14}$$

$$+ \left(\frac{9}{14} FG + \frac{4}{18} EH + \frac{6}{50} DI - BJ \right) \left(\frac{\lambda}{a}\right)^{15} +$$

$$+ \left(\frac{1}{16} G^2 + \frac{24}{7} FH + \frac{1}{120} EI - \frac{1}{2} CJ - \frac{4}{7} AK \right) \left(\frac{\lambda}{a}\right)^{16} + \left(\frac{2}{3} GH + \frac{9}{70} FI - \frac{12}{5} DJ - \frac{4}{7} BK \right) \left(\frac{\lambda}{a}\right)^{17}$$

$$+ \left(\frac{16}{9} H^2 + \frac{1}{40} GI - \frac{1}{6} EJ - \frac{2}{7} CK \right) \left(\frac{\lambda}{a}\right)^{18} + \left(\frac{4}{30} HI - \frac{18}{7} FJ - \frac{48}{35} DK \right) \left(\frac{\lambda}{a}\right)^{19}$$

$$+ \left(\frac{1}{400} I^2 - \frac{1}{2} GJ - \frac{2}{21} EK \right) \left(\frac{\lambda}{a}\right)^{20} + \left(- \frac{8}{3} HJ - \frac{72}{49} FK \right) \left(\frac{\lambda}{a}\right)^{21} + \left(- \frac{1}{10} IJ - \frac{2}{7} GK \right) \left(\frac{\lambda}{a}\right)^{22}$$

$$+ \left(J^2 - \frac{4}{70} IK \right) \left(\frac{\lambda}{a}\right)^{-4} + \frac{8}{7} JK \left(\frac{\lambda}{a}\right)^{26} + \frac{16}{49} K^2 \left(\frac{\lambda}{a}\right)^{28} - \frac{32}{21} HK \left(\frac{\lambda}{a}\right)^{23}$$

$$+ \Big\{ \frac{9}{100} A_2^2 \left(\frac{\lambda}{a}\right)^6 + \frac{1}{10} A_2 (2A_3 - A_1) \left(\frac{\lambda}{a}\right)^7 + \Big[\frac{1}{36} (2A_3 - A_1)^2 - \frac{9}{2} A_2 \lambda_2 \Big] \left(\frac{\lambda}{a}\right)^8$$

$$+\left\{144 h_2^2 \left(\frac{r}{a}\right)^4 + \frac{36}{5} f_2 h_2 \left(\frac{r}{a}\right)^5 + \left[\frac{9}{100} f_2^2 + 4(2f_3 - f_1) h_2\right]\left(\frac{r}{a}\right)^6 + \frac{1}{10} f_2 (2f_3 - f_1)\left(\frac{r}{a}\right)^7\right.$$

$$+\left[\frac{1}{36}(2f_3 - f_1)^2 - \frac{9}{2} f_2 h_2\right]\left(\frac{r}{a}\right)^8 - \frac{9}{80} f_2^2 \left(\frac{r}{a}\right)^9 - \left[\frac{1}{16} f_2 (2f_3 - f_1) + \frac{24}{5} f_3 h_3\right]\left(\frac{r}{a}\right)^{10}$$

$$\left. - \frac{6}{50} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \left[\frac{9}{25} f_2^2 - \frac{1}{15} f_3 (2f_3 - f_1)\right]\left(\frac{r}{a}\right)^{12} + \frac{3}{40} f_2 f_3 \left(\frac{r}{a}\right)^{14} + \frac{1}{25} f_3^2 \left(\frac{r}{a}\right)^{16}\right\}$$

$$\int_0^{2\pi} \widehat{rr}\cdot\widehat{\theta\theta}\, d\theta = \pi E^2 \left(\frac{a}{R}\right)^4 \left[2\left\{\frac{1}{4} q_0^2 + \frac{1}{4} A q_0 \left(\frac{r}{a}\right)^2 + \frac{1}{4} B q_0 \left(\frac{r}{a}\right)^3 + \left(\frac{3}{64} A^2 + \frac{1}{8} C q_0\right)\left(\frac{r}{a}\right)^4\right.\right.$$

$$+\left(\frac{7}{80} AB + \frac{3}{5} D q_0\right)\left(\frac{r}{a}\right)^5 + \left(\frac{4}{100} B^2 + \frac{1}{24} AC + \frac{1}{24} E q_0\right)\left(\frac{r}{a}\right)^6 + \left(\frac{3}{80} BC + \frac{27}{140} AD + \frac{9}{14} F q_0\right)\left(\frac{r}{a}\right)^7$$

$$+\left(\frac{5}{24^2} C^2 + \frac{6}{35} BD + \frac{5}{48\times 8} AE + \frac{1}{8} G q_0\right)\left(\frac{r}{a}\right)^8 + \left(\frac{11}{140} CD + \frac{11}{960} BE + \frac{11}{56} AF + \frac{12}{18} H q_0\right)\left(\frac{r}{a}\right)^9$$

$$+\left(\frac{216}{35^2} D^2 + \frac{1}{192} CE + \frac{12}{70} BF + \frac{3}{80} AG + \frac{1}{40} I q_0\right)\left(\frac{r}{a}\right)^{10} + \left(\frac{13}{35\times 16} DE + \frac{13}{24\times 7} CF + \frac{13}{400} BG + \frac{13}{66} AH\right)\left(\frac{r}{a}\right)^{11}$$

$$+\left(\frac{7}{96} E^2 + \frac{12}{35} DF + \frac{7}{480} CG + \frac{98}{165} BH + \frac{7}{960} AI - \frac{1}{2} J q_0\right)\left(\frac{r}{a}\right)^{12} +$$

$$+\left(\frac{5}{32\times 7} EF + \frac{9}{140} DG + \frac{5}{66} CH + \frac{1}{160} BI\right)\left(\frac{r}{a}\right)^{13} + \left(\frac{1}{49} F^2 + \frac{1}{240} EG + \frac{134}{385} DH + \frac{1}{360} CI - \frac{1}{7} AJ - \frac{2}{7} K q_0\right)\left(\frac{r}{a}\right)^{14}$$

$$+\left(\frac{17}{960} FG + \frac{17}{33\times 24} EH + \frac{17}{1400} DI - \frac{17}{140} BJ\right)\left(\frac{r}{a}\right)^{15} +$$

388

$$+\left(\frac{9}{1600}G^2+\frac{24}{77}FH+\frac{1}{1280}EI-\frac{3}{56}CJ-\frac{9}{112}AK\right)\left(\frac{r}{a}\right)^{16}$$

$$+\left(\frac{19}{330}GH+\frac{19}{7\times 140}FI-\frac{57}{245}DJ-\frac{19}{280}BK\right)\left(\frac{r}{a}\right)^{17}$$

$$+\left(\frac{160}{1089}H^2+\frac{1}{480}GI-\frac{5}{24\times 14}EJ-\frac{5}{24\times 7}CK\right)\left(\frac{r}{a}\right)^{18}$$

$$+\left(\frac{7}{660}HI-\frac{3}{14}FJ-\frac{9}{70}DK\right)\left(\frac{r}{a}\right)^{19}+\left(\frac{11}{240}I^2-\frac{11}{480}GJ-\frac{11}{14\times 16}EK\right)\left(\frac{r}{a}\right)^{20}$$

$\frac{11}{280}$

$$+\left(-\frac{46}{231}HJ-\frac{23}{28\times 7}FK\right)\left(\frac{r}{a}\right)^{21}+\left(-\frac{1}{140}IJ-\frac{3}{140}GK\right)\left(\frac{r}{a}\right)^{22}$$

$$+\left(\frac{13}{14^2}J^2-\frac{13}{120\times 28}IK\right)\left(\frac{r}{a}\right)^{24}+\frac{1}{14}JK\left(\frac{r}{a}\right)^{26}+\frac{15}{28^2}K^2\left(\frac{r}{a}\right)^{28}-\frac{25}{231}HK\left(\frac{r}{a}\right)^{23}$$

$$+\Big\{-4p_2^2-24p_2r_2\left(\frac{r}{a}\right)^2-\frac{19}{35}f_2p_2\left(\frac{r}{a}\right)^3-\frac{7}{24}(2f_3-f_1)p_2\left(\frac{r}{a}\right)^4+\left[\frac{13}{40}f_2r_2+\frac{20}{4900}f_2^2+\frac{7}{8}(2f_3-f_1)r_2\right]\left(\frac{r}{a}\right)^6$$

$$+\frac{6}{35}f_3r_2\left(\frac{r}{a}\right)^5$$

$$+\frac{1}{192}f_2(2f_3-f_1)\left(\frac{r}{a}\right)^7+\left[\frac{15}{16^2}(2f_3-f_1)^2-\frac{3}{20}f_2r_2+\frac{7}{20}f_3p_2\right]\left(\frac{r}{a}\right)^8$$

$$-\frac{17}{2800}f_2^2\left(\frac{r}{a}\right)^9-\left[\frac{29}{96\times 80}f_2(2f_3-f_1)+\frac{3}{20}f_3r_2\right]\left(\frac{r}{a}\right)^{10}-\frac{1}{160}f_2f_3\left(\frac{r}{a}\right)^{11}$$

$$+\left[\frac{14}{6400}f_2^2-\frac{1}{8\times 32}f_3(2f_3-f_1)\right]\left(\frac{r}{a}\right)^{12}+\frac{29}{6400}f_2f_3\left(\frac{r}{a}\right)^{14}+\frac{15}{6400}f_3^2\left(\frac{r}{a}\right)^{16}\Big\}$$

390

$$\int_0^{2\pi} r\dot{\theta}^2\, d\theta = E^2\left(\frac{a}{R}\right)^4 \pi \Big[4p_2^2 + 24 p_2 r_2 \left(\frac{r}{a}\right)^2 + \frac{16}{35} f_2 p_2 \left(\frac{r}{a}\right)^3 + \left\{ 36 r_2^2 + \frac{5}{24}(2f_3 - f_1) p_2 \right\}\left(\frac{r}{a}\right)^4$$

$$+ \frac{48}{35} f_2 r_2 \left(\frac{r}{a}\right)^5 + \left\{ \frac{64}{4900} f_2^2 + \frac{5}{8}(2f_3 - f_1) r_2 \right\}\left(\frac{r}{a}\right)^6 + \frac{5}{420} f_2 (2f_3 - f_1)\left(\frac{r}{a}\right)^7$$

$$+ \left\{ \frac{25}{96^2}(2f_3 - f_1)^2 - \frac{10.5}{20} f_2 r_2 - \frac{3}{20} f_3 p_2 \right\}\left(\frac{r}{a}\right)^8 - \frac{1}{100} f_2^2 \left(\frac{r}{a}\right)^9 - \Big\{ \frac{7}{96\times 16} f_2 (2f_3 - f_1)$$

$$+ \frac{9}{20} f_3 r_2 \Big\}\left(\frac{r}{a}\right)^{10} - \frac{3}{350} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \left[\frac{3.5^2}{80^2} f_2^2 - \frac{3}{8\times 96} f_3 (2f_3 - f_1) \right]\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{21}{80^2} f_2 f_3 \left(\frac{r}{a}\right)^{14} + \frac{9}{6400} f_3^2 \left(\frac{r}{a}\right)^{16} \Big]$$

The part from non-uniform part

$$\int_0^{2\pi} \left[r\dot{\theta} \cdot c\dot{\theta} - r\dot{\theta}^2 \right] d\theta = E^2\left(\frac{a}{R}\right)^4 \pi \Big[-8p_2^2 - 48 p_2 r_2 \left(\frac{r}{a}\right)^2 - f_2 p_2 \left(\frac{r}{a}\right)^3 - \left\{ 36 r_2^2 + \frac{1}{2}(2f_3 - f_1) p_2 \right\}\left(\frac{r}{a}\right)^4$$

$$- \frac{6}{5} f_2 r_2 \left(\frac{r}{a}\right)^5 - \left\{ \frac{44}{4900} f_2^2 - \frac{1}{2} f_2 r_2 + \frac{1}{24}(2f_3 - f_1) r_2 \right\}\left(\frac{r}{a}\right)^6 - \frac{3}{448} f_2 (2f_3 - f_1)\left(\frac{r}{a}\right)^7$$

$$- \left\{ \frac{10}{96^2}(2f_3 - f_1)^2 - \frac{7.5}{20} f_2 r_2 - \frac{1}{2} f_3 p_2 \right\}\left(\frac{r}{a}\right)^8 + \frac{11}{2800} f_2^2 \left(\frac{r}{a}\right)^9 + \Big\{ \frac{1}{12800} f_2 (2f_3 - f_1) \qquad \frac{1}{1280}$$

$$+ \frac{3}{10} f_3 r_2 \Big\}\left(\frac{r}{a}\right)^{10} + \frac{13}{350\times 16} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \frac{1.75}{6400} f_2^2 \left(\frac{r}{a}\right)^{12} + \frac{1}{800} f_2 f_3 \left(\frac{r}{a}\right)^{14} + \frac{3}{3200} f_3^2 \left(\frac{r}{a}\right)^{16}$$

39

$$\frac{\dfrac{\mathcal{C}_1}{R^3}}{\dfrac{\pi E}{2}\left(\dfrac{A}{R}\right)^6\left(\dfrac{t}{R}\right)} = 0.35\,q_0^2 + 0.175\,A q_0 + 0.140\,B q_0 + (0.04270833\,A^2 + 0.0583333\,C q_0) + (0.0778571\,AB$$

$$+ 0.24000\,D q_0) + (0.0365000\,B^2 + 0.03541667\,AC + 0.0145833\,E q_0) + (0.03388889\,BC + 0.1552381\,AD$$

$$+ 0.20000\,F q_0) + (0.00798611\,C^2 + 0.150857143\,BD + 0.009895833\,AE + 0.036000\,G q_0) +$$

$$+ (0.07194805\,CD + 0.009734848\,BE + 0.14090909\,AF + 0.16969696\,H q_0) + (0.1635718\,D^2 +$$

$$+ 0.00468875\,CE + 0.1400000\,BF + 0.02541667\,AG + 0.005833333\,I q_0) + (0.021483516\,DE +$$

$$+ 0.067948718\,CF + 0.0254615\,BG + 0.1263403\,AH) + (0.00070995\,E^2 + 0.3134694\,DF$$

$$+ 0.0127405\,CG + 0.1274459\,BH + 0.00443452\,AI - 0.100000\,J q_0) + (0.02083333\,EF +$$

$$+ 0.051771439\,DG + 0.0626262\,CH + 0.0045000\,BI) + (0.1535714\,F^2 + 0.003854166\,EG +$$

$$+ 0.2117481\,DH + 0.00922222\,CI - 0.07857143\,AJ - 0.050000\,K q_0) + (0.0570588\,FG + 0.0195781\,EH$$

$$+ 0.0104033\,DI - 0.0805042\,BJ) + (0.00531447\,G^2 + 0.29090909\,FH + 0.00070093\,EI$$

0.01043759651

$$- 0.04007437\,CJ - 0.04027777\,AK) + (0.0544176\,GH + 0.0100626\,FI - 0.1889581\,DJ - 0.0415789\,BK)$$

$$+ (0.1395776\,H^2 + 0.00125833\,GI - 0.0127762\,EJ - 0.02083333\,CK)$$

392

$$+\left(0.01007215\,HI - 0.1918367\,FJ - 0.00987755\,DK\right) + \left(0.0001821338\,I^2 - 0.04003788\,GJ\right.$$

$$0.0361688511$$

$$\left. - 0.00672348\,EK\right) - 0.1868624\,HJ - 0.10133097\,FK - 0.00678571\,IJ - 0.01916667\,GK$$

$$+ 0.06365777\,J^2 - 0.00362177\,IK + 0.06836735\,JK + 0.01845238\,K^2 - 0.09939394\,HK$$

$$+\left\{10.4\,p_2^2 + 31.2\,p_2 r_2 + 0.52\,f_3 p_2 + 39.6\,r_2^2 + 0.21666667\,(2f_3 - f_1)p_2 + 1.47428571\,f_2 r_2\right.$$

$$+ 0.01416837\,f_2^2 - 0.162500\,f_2 p_2 + 0.662500\,(2f_3 - f_1)r_2 + 0.01304563\,f_2(2f_3 - f_1)$$

$$0.00537760$$

$$+ 0.003059896\,(2f_3 - f_1)^2 - 0.547500\,f_3 r_2 - 0.13000\,f_3 p_2 - 0.01115844\,f_2^2 - 0.00522526\,f_2(2f_3 - f_1)$$

$$- 0.465\,f_3 r_2 - 0.00969505\,f_2 f_3 + 0.002460379\,f_2^2 - 0.004761905\,f_3(2f_3 - f_1) + 0.004484375\,f_2 f_3$$

$$\left. + 0.002086806\,f_3^2\right\}$$

393

$$2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2$$

$$= -2.25000f_1 - 1.687500f_2 + 9f_1^2 + 6.750000f_1f_2 - 4.5000f_2^2$$

$$f_2(f_1 - 2f_3) = 2.250000f_1f_2 + 1.6875000f_2^2$$

$$12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2$$

$$= -3f_2 - 20.25000f_1^2 - 18.37500f_1f_2 - 11.3906250f_2^2$$

$$4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2 = 0.6250f_1 + 0.84375f_2 - 2.5000f_1^2 - 10.12500f_1f_2 - 5.062500f_2^2$$

$$f_2f_3 = -0.625000f_1f_2 - 0.84375f_2^2$$

$$32f_3^2 - 16f_1f_3 - 9f_2^2 = 16f_3(2f_3 - f_1) - 9f_2^2$$

$$= 16(0.625f_1 + 0.84375f_2)(2.2500f_1 + 1.68750f_2) - 9f_2^2$$

$$= 22.500f_1^2 + 47.2500f_1f_2 + 13.781250f_2^2$$

$$f_3^2 = 0.390625f_1^2 + 1.0546875f_1f_2 + 0.7119140625f_2^2$$

$$f_1(1 - 2f_1) = f_1 - 2f_1^2$$

$$f_2(1 - 4f_1) = f_2 - 4f_1f_2$$

To calculate the bending energy: 3RS

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{ 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$= \frac{1}{R}\left\{ 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$k_1 + k_2 = \frac{1}{R}\left\{ 4f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 9f_2\left[1-2\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 16f_3\left[1-2\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$= \frac{1}{R}\left\{ 4f_1 + 9f_2\left(\frac{r}{a}\right) + 8(2f_3 - f_1)\left(\frac{r}{a}\right)^2 - 18f_2\left(\frac{r}{a}\right)^4 - 32f_3\left(\frac{r}{a}\right)^6 \right\}$$

$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2 \left\{ 8f_1^2 + 24f_1f_2 + \frac{81}{4}f_2^2 + 16f_1(2f_3-f_1) + \frac{144}{5}f_2(2f_3-f_1) + \frac{32}{3}(2f_3-f_1)^2 - 24f_1f_2 - \frac{324}{7}f_2^2 - 32f_1f_3 - 36f_2(2f_3-f_1) - 64f_2f_3 + 32.4f_2^2 - 51.2f_3(2f_3-f_1) + 96f_2f_3 + \frac{512}{7}f_3^2 \right\}$$

$$\int_0^a (k_1k_2) r\,dr = \left(\frac{a}{R}\right)^2 \left\{ 2f_1^2 + 6f_1f_2 + \frac{9}{2}f_2^2 + 4f_1(2f_3-f_1) + 6f_2(2f_3-f_1) + 2(2f_3-f_1)^2 - 6f_2f_1 - 9f_2^2 - 6f_2(2f_3-f_1) - 8f_1f_3 - 12f_2f_3 + 4.5f_2^2 - 8f_3(2f_3-f_1) + 12f_2f_3 + 8f_3^2 \right\}$$

$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 8f_1^2 + 0 + 6.364286 f_2^2 + 16 f_1(2f_3 - f_1)$$

$$- 7.2 f_2(2f_3-f_1) + \frac{32}{3}(2f_3-f_1)^2 - 32 f_1 f_3 + 32 f_2 f_3 - 51.2 f_3(2f_3-f_1)$$

$$+ \frac{512}{7} f_3^2 \Big\}$$

$$\int_0^a (k_1 k_2)\, r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 2f_1^2 + 0 + 0 + 4f_1(2f_3-f_1) + 0$$

$$+ 2(2f_3-f_1)^2 - 8f_1 f_3 + 0 - 8f_3(2f_3-f_1) + 8f_3^2 \Big\}$$

$$\frac{\mathcal{E}_2}{R^3} = \frac{1}{12}\left(\frac{t}{R}\right)^3 \frac{E\pi}{(1-\nu^2)}\left(\frac{a}{R}\right)^2 \Big\{ 2.8 f_1^2 + 6.364286 f_2^2 + 5.6(2f_3-f_1)f_1$$

$$- 7.2 f_2(2f_3-f_1) + 5.4666667(2f_3-f_1)^2 - 11.2 f_1 f_3 + 32 f_2 f_3$$

$$- 30.4 f_3(2f_3-f_1) + 52.34285714 f_3^2 \Big\}$$

$$\frac{\dfrac{\mathcal{E}_2}{R^3}}{\dfrac{\pi E}{2}\left(\dfrac{a}{R}\right)^2\left(\dfrac{t}{R}\right)^3} = 0.5128205 f_1^2 + 1.1656201 f_2^2 + 1.0256410 f_1(2f_3-f_1)$$

$$- 1.3186813 f_2(2f_3-f_1) + 1.0012210(2f_3-f_1)^2$$

$$- 2.0512821 f_1 f_3 + 5.8608059 f_2 f_3 - 5.5677656 f_3(2f_3-f_1)$$

$$+ 9.5866039 f_3^2$$

$$\frac{\mathcal{E}_1/R^3}{\frac{1}{2}\frac{\sigma^2}{E}\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2} = 8s_2^2 + 2(1+\nu)\left\{(r_0^2+2s_2^2)+12\,s_2q_2+12q_2^2\right\}$$

$$\frac{\delta\theta/R^3}{\frac{\sigma^2}{2E}\pi\left(\frac{t}{R}\right)\left(\frac{a}{R}\right)^2} = 4s_2 - 2(1+\nu)\,r_0$$

For the region outside the circle:

$$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos2\theta\left\{\frac{1}{2} - \frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{4s_2}{\left(\frac{r}{a}\right)^2}\right\}\right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos2\theta\left\{\frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{1}{2}\right\}\right]$$

$$\widehat{r\theta} = -\sigma\sin2\theta\left\{\frac{1}{2} + \frac{6q_2}{\left(\frac{r}{a}\right)^4} + \frac{2s_2}{\left(\frac{r}{a}\right)^2}\right\}$$

$$\frac{u}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[\frac{1}{2}(1-\nu)\left(\frac{r}{a}\right) - (1+\nu)\frac{r_0}{\left(\frac{r}{a}\right)} + \cos2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{a}\right) + 2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} + \frac{4s_2}{\left(\frac{r}{a}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{a}\right)\right]\sin2\theta$$

$$\frac{\mathcal{E}_2/R^3}{\frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3} = 0.5128205 f_1^2 + 1.1656901 f_2^2 - 1.0256410 f_1 (2.2500 f_1 + 1.6875 f_2)$$

$$+ 1.3186813 f_2 (2.2500 f_1 + 1.6875 f_2) + 1.0012210 (2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 2.0512821 f_1 (0.625 f_1 + 0.84375 f_2) - 5.8608059 f_2 (0.625 f_1 + 0.84375 f_2)$$

$$- 5.5677156 (0.625 f_1 + 0.84375 f_2)(2.250 f_1 + 1.6875 f_2) + 9.5866039 (0.625 f_1 + 0.84375 f_2)^2$$

=

f_1^2	$f_1 f_2$	f_2^2
+0.5128205		+1.1656901
−2.3076923	−1.7307692	
	+2.9670329	+2.2252747
+5.0686813	+7.6030220	+2.8511332
+1.2820513	+1.7307693	
	−3.6630037	−4.9450550
−7.8296704	−16.4423078	−7.9275413
+3.7447671	+10.1108713	+6.8248381
+0.4709575	+0.5756148	+0.1942698

$$\mathcal{E}_2/R^3 = \frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3 \left\{ 0.4709575 f_1^2 + 0.5756148 f_1 f_2 + 0.1942698 f_2^2 \right\}$$

$q_0 - \frac{1}{7} =$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
−0.500000		+1.00000		
	−0.50000		+2.00000	
+0.562500	+0.421875	−2.25000	−1.6875	+1.125000
			−2.7000	−2.025000
	+0.250000	+1.6875	+1.53125	+0.94921875
				−1.28571429
−0.15625	−0.2109375	+0.625	+2.53125	+1.265625
			+0.2083333	+0.2812500
		−1.125	−2.3625	−0.6890625
		+0.22321429	+0.60267857	+0.40680804

$-0.093750 f_1 - 0.03906250 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 \pm 0.02812500 f_2^2$

0.12351183

$\underline{\underline{400}}$

$r_0 =$ f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
−0.1250		+0.2500		
	−0.150		+0.60000	
+0.18750	+0.140625	−0.75000	−0.562500	+0.37500
			−0.96428571	−0.72321429
	+0.09375	+0.6328125	+0.5742[illegible]	+0.3559[illegible]
				−0.5000
−0.0625	−0.084375	+0.25	+1.0125	+0.50625
			+0.07305195	+0.09862013
		−0.46875	−0.984375	−0.2826[illegible]
		+0.09765625	+0.2636[illegible]	+0.1279[illegible]

$$\eta\left\{0 + 0 + 0.0112[illegible]\, f_1^2 + 0.012[illegible]\, f_1 f_2 + 0.0034[illegible]\, f_2^2\right\}$$

$$r_0^2 = \eta^2\left\{0.0001373291\, f_1^4 + 0.0002878563\, f_1^3 f_2 + 0.0002324539\, f_1^2 f_2^2 + 0.0000855334\, f_1 f_2^3 + 0.0000121244\, f_2^4\right\}$$

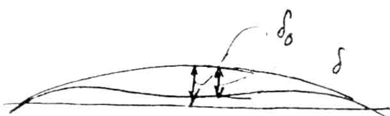
$$r_{f_2}^2 = \eta^2\left\{0.0000203455\, f_1^2 + 0.0000235440\, f_1 f_2 + 0.0000068113\, f_2^2\right\}$$

$$\mathcal{E}_3/R^3 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\Big\{ 0.0003570557 f_1^4 + 0.0007484264 f_1^3 f_2$$

$$+ 0.0006043801 f_1^2 f_2^2 + 0.0002223868 f_1 f_2^3 \pm 0.0000315234 f_2^4$$

$$+ 0.0000528943 f_1^2 + 0.0000612144 f_1 f_2 + 0.0000177094 f_2^2 \Big\}\eta^2$$

$$\mathcal{D}/R^3 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\eta\Big\{ -0.03046875 f_1^2 - 0.03193286 f_1 f_2 - 0.00905323 f_2^2 \Big\}$$



$$\frac{\delta}{\delta_0} = f_1 + f_2 + f_3$$

$$= 0.375 f_1 + 0.15625 f_2$$

The terms in { } in p. 393 can be collected as

$$\eta^2\Big\{ 10.4\, p_2^2 + 31.2\, p_2 r_2 + 0.3575\, f_2 p_2 + 39.6\, r_2^2 - 0.21666667\, p_2 (2.250 f_1 + 1.6875 f_2)$$

$$+ 0.9267857\, f_2 r_2 + 0.00547291\, f_2^2 - 0.6625000\,(2.2500 f_1 + 1.6875 f_2)\, r_2$$

$$- 0.00766803\, f_2\,(2.2500 f_1 + 1.6875 f_2) + 0.003059896\,(2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 0.13000\, p_2\,(0.625 f_1 + 0.84375 f_2) + 0.465\, r_2\,(0.625 f_1 + 0.84375 f_2)$$

$$+ 0.00521067\, f_2\,(0.625 f_1 + 0.84375 f_2) - 0.004761905\,(0.625 f_1 + 0.84375 f_2)$$

$$(2.25 f_1 + 1.6875 f_2) + 0.002086806\,(0.625 f_1 + 0.84375 f_2)^2 \Big\}$$

$= 0.6500000 +$

$$+\eta\left\{\left(\overset{0.0609378}{0.0058594\,f_1} + \overset{0.0253906}{0.00244140\,f_2}\right)\times 10.4 - 0.1625000\,f_1 - 0.089375\,f_2\right.$$

$$\left.+\ 0.121875\,f_1 + 0.09140625\,f_2 - 0.0203125\,f_1 - 0.02742187 5\,f_2\right\}$$

$\rightarrow$ (0 + 0)

$+\eta^2\Big\{$

f_1	f_2	f_1^2	f_1f_2	f_2^2
		+0.00142824	+0.00119019	+0.00024795
		−0.00761722	−0.00317382	
			−0.00418947	−0.00174560 −0.00160260
		+0.0171875		
			+0.01930804	
				+0.00542291
		−0.03105469	−0.02329102	
			−0.01725307	−0.01293980
		+0.01549072	+0.02323609	+0.00871353
		+0.00605469	+0.00817383	
			+0.00325667	+0.00439650
		−0.0011205	−0.00687865	−0.00404481
		−0.00683343	−0.01351121	−0.00578621
		$-0.00534419\,f_1^2$	$-0.00625377\,f_1f_2$	$-0.00149772\,f_2^2$
		+0.00036874	+0.0037879	+0.00010068

Independent check

η^2	f_1^2	$f_1 f_2$	f_2^2
	+0.00142824	+0.00119019	+0.00027295
	−0.00761722	−0.00317582	
		−0.00418947	−0.00174560
	+0.0171875		
		+0.01930804	
			+0.00542291
	−0.00985104	−0.01064295	−0.00244100
	−0.00077874	−0.00211321	−0.00143358
	+0.00036874	+0.00037878	+0.00010068

$$A = f_1 - 2f_1^2$$

$$B = f_2 - 4f_1 f_2$$

$$C = -2.25000 f_1 - 1.6875 f_2 + 9 f_1^2 + 6.750000 f_1 f_2 - 4.50000 f_2^2$$

$$D = 2.25000 f_1 f_2 + 1.687500 f_2^2$$

$$E = -3f_2 - 20.25000 f_1^2 - 18.37500 f_1 f_2 - 11.3906250 f_2^2$$

$$F = f_2^2$$

$$G = 0.625 f_1 + 0.84375 f_2 - 2.5000 f_1^2 - 10.12500 f_1 f_2 - 5.0625000 f_2^2$$

$$H = -0.6250 f_1 f_2 - 0.84375 f_2^2 = J$$

$$I = 22.500 f_1^2 + 47.2500 f_1 f_2 + 13.78125 f_2^2$$

$$K = 0.390625 f_1^2 + 1.0546875 f_1 f_2 + 0.7119140625 f_2^2$$

$$\eta q_0 = 1 + \eta \{ -0.09375 f_1 - 0.0390625 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2 \}$$

$$\eta^2 q_0^2 = 1 + \eta \{ -0.1875 f_1 - 0.078125 f_2 + 0.32142858 f_1^2 + 0.24702380 f_1 f_2 + 0.0562500 f_2^2 \}$$

$$+ \eta^2 \{ +0.0087890625 f_1^2 + 0.0073242187 5 f_1 f_2 + 0.0015258789 1 f_2^2$$
$$- 0.03013393 f_1^3 - 0.03571429 f_1^2 f_2 - 0.01492280 f_1 f_2^2 - 0.00219727 f_2^3$$
$$+ 0.02582908 f_1^4 + 0.03970025 f_1^3 f_2 + 0.02429537 f_1^2 f_2^2 + 0.00694754 f_1 f_2^3$$
$$+ 0.000791016 f_2^4 \}$$

X 0.35 !!!

$$\eta^3 q_0 \Big[0.175A + 0.140B + 0.05833333C + 0.24D + 0.01458333E + 0.2F$$
$$+ 0.035G + \overset{0.069696967}{\cancel{0.16969696}}\,H + 0.00583333I - 0.1J - 0.05K \Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
0.175		−0.35		
	0.14		−0.56	
−0.13125	−0.0984375	+0.525	+0.39375	−0.2625
			+0.54	+0.405
	−0.04375	−0.2953125	−0.2679687 5	−0.16611328
				+0.2
+0.021875	+0.02953125	−0.0875	−0.354375	−0.1771875
			−0.04356060 6	−0.05880682
		+0.13125	+0.275625	+0.080390[illegible]
		−0.01953125	−0.052734375	−0.035595703125

$$(+0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2)\,\eta$$

$$\eta q_0 = 1 + \eta\{-0.09375 f_1 - 0.0390625 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2\}$$

$$= \eta\{0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2\}$$

$$- \eta^2 \{ 0.00615234 f_1^2 + 0.00512695 f_1 f_2 + 0.00106812 f_2^2 - 0.01955566 f_1^3 -$$
$$\boxed{\begin{matrix} -0.02274714 \\ -0.01631860 f_1^2 f_2 \end{matrix}} - 0.00931728 f_1 f_2^2 - 0.00134766 f_2^3 + 0.01544364 f_1^4 +$$
$$+ 0.02300039 f_1^3 f_2 + 0.01363814 f_1^2 f_2^2 + 0.00377758 f_1 f_2^3 + 0.00041661 f_2^4 \}$$

405

$\eta^2 A[0.04270833A + 0.0778571B + 0.03541667C + 0.1552381D + 0.00989583E$
$+ 0.14090909F + 0.02541667G + \overset{0.0477689}{0.1263403}H + 0.00443452I - 0.02857143J$
$- 0.04027777K]$

f_1	f_2	f_1^2	f_1f_2	f_2^2
+0.04270833		−0.08541667		
	+0.0778571		−0.3114284	
−0.0796875	−0.059765625	+0.31875	+0.2390625	−0.159375
			+0.3492857	+0.2619643
	−0.0296875	−0.200390625	−0.1818359	−0.1127197
				+0.1409091
+0.0158854	+0.0214453	−0.0635417	−0.2573437	−0.1286719
			−0.0298556	−0.0403050
		+0.0997767 +0.0997317	+0.2095311 +0.2094366	+0.0611132 +0.0610856
		−0.0157335	−0.0424805	−0.0286743

$(-0.0210938f_1 + 0.0098493f_2 + 0.0533992f_1^2 - 0.0251594f_1f_2 - 0.0057869f_2^2)\eta$

$+0.0534442 \quad -0.0250649 \quad -0.0057587$

$-(-f_1 + 2f_1^2)\eta^2$

$= -\eta^2\{+0.0210938f_1^2 - 0.0098493f_1f_2 \overset{-0.0956318}{- 0.0955868}f_1^3 \overset{+0.0447635}{+ 0.0448580}f_1^2f_2$

$+ 0.0057869f_1f_2^2 + 0.1067984f_1^4 - 0.0503188f_1^3f_2 - 0.0115738f_1^2f_2^2\}$

$+0.0057587 \quad +0.1068884 \quad -0.0501298 \quad -0.0115174$

406

$\eta^2 B[0.0365 B + 0.03388889 C + 0.150857143 D + 0.009734848 E + 0.14 F$

$+ 0.0254615 G +$ ~~0.1274459~~ 0.0469417 $H + 0.0045 I -$ ~~0.0805042 J~~ $- 0.0415789 K]$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	+0.0365		−0.146	
−0.07625	−0.0571875	+0.305	+0.22875	−0.1525
			+0.3394286	+0.2545714
	−0.0292045	−0.1971307	−0.1788778	−0.1108860
				+0.14
+0.0159134	+0.0214831	−0.0636538	−0.2577977	−0.1288988
			−0.0293386	−0.0396071
		+0.10125	+0.212625	+0.062015625
		−0.0162418	−0.0438527	−0.0296006

$$-0.0603366 f_1 - 0.0284089 f_2 + 0.1292237 f_1^2 + 0.1249368 f_1 f_2 - 0.0049055 f_2^2$$

$$0 \quad + \quad f_2 \quad + \quad 0 \quad - 4 \quad f_1 f_2 \quad + \quad 0$$

$$\eta^2 \{ -0.0603366 f_1 f_2 - 0.0284089 f_2^2 + 0.3705701 f_1^2 f_2 + 0.2385724 f_1 f_2^2$$
$$- 0.0049055 f_2^3 - 0.5168948 f_1^3 f_2 - 0.4997472 f_1^2 f_2^2 + 0.0196220 f_1 f_2^3 \}$$

$$C[0.007986111\,C + 0.07194805\,D + 0.0046875\,E + 0.067948718\,F + 0.0124405\,G + 0.0225468\overset{3}{9}\,H + 0.00222222\,I - 0.020833333\,K]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
−0.0179687	−0.0134766	+0.071875	+0.05390625	−0.0359375
			+0.1618831	+0.1214123
	−0.0140625	−0.0949219	−0.0861328	−0.0533936
				+0.0679487
+0.0077753	+0.0104967	−0.0311013	−0.1259601	−0.0622800
			−0.0140918	−0.0190239
		+0.05	+0.105	+0.030625
		−0.00813802	−0.0219766	−0.01483154

$$\frac{-0.0101934 f_1 - 0.0170424 f_2 - 0.0122862 f_1^2 + 0.0726390 f_1 f_2 + 0.0338195 f_2^2}{-2.25 f_1 - 1.6875 f_2 + 9 f_1^2 + 6.25 f_1 f_2 - 4.5 f_2^2}$$

$$= \eta^2 \{ 0.0229352 f_1^2 + 0.0555468 f_1 f_2 + 0.0287591 f_2^2 - 0.0640967 f_1^3 - \overset{0.3643761}{0.0995788} f_1^2 f_2 - 0.2678263 f_1 f_2^2 + 0.0196204 f_2^3 - 0.1105758 f_1^4 + 0.5707562 f_1^3 f_2 + 0.8499294 f_1^2 f_2^2 - 0.0985624 f_1 f_2^3 - 0.1521878 f_2^4 \}$$

$$D\left[0.1635918\,D + 0.021483516\,E + 0.3134694\,F + 0.05771429\,G + 0.1029900\,H + 0.0104033\,I - 0.00987755\,K\right]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
			+0.3680816	+0.2760612
	−0.0644505	−0.4350412	−0.3947596	−0.2447107
				+0.3134694
+0.0360714	+0.0486964	−0.1442857	−0.5843572	−0.2921786
			−0.0643688	−0.0868978
		+0.234075	+0.4915575	+0.1433709
		−0.0038584	−0.0104177	−0.0070320

$$+0.0360714\,f_1 - 0.0157541\,f_2 - 0.3491103\,f_1^2 - 0.1942642\,f_1 f_2 + 0.1020824\,f_2^2$$

$$+\ 2.25\ f_1 f_2 + 1.6875\ f_2^2$$

$$\eta^2\left\{+0.0811607\,f_1^2 f_2 + 0.0254238\,f_1 f_2^2 - 0.0265850\,f_2^3 - 0.7854982\,f_1^3 f_2 - 1.0262181\,f_1^2 f_2^2 - 0.0981354\,f_1 f_2^3 + 0.1722641\,f_2^4\right\}$$

$$E\Big[\,0.00070995\,E + 0.0208333\,F + 0.0038541667\,G + 0.0067805\,H + 0.00070023\,I - 0.00672348\,K\,\Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	-0.0021299	-0.0143765	-0.0130453	-0.00808677
				$+0.0208333$
$+0.0024089$	$+0.0032520$	-0.0096354	-0.0390234	-0.0195117
			-0.0042378	-0.0057210
		$+0.0157552$	$+0.0330859$	$+0.0096509\,45$ $+0.0097131$
		-0.0026264	-0.0070912	-0.0047865
$+0.0024089\,f_1$	$+0.0011221\,f_2$	$-0.0108831\,f_1^2$	$-0.0303118\,f_1 f_2$	-0.0076226 $-0.0075596\,f_2^2$

$$-3 \quad f_2 - 20.25 \quad f_1^2 - 18.375 \quad f_1 f_2 - 11.390625\,f_2^2$$

$$\eta^2\Big\{-0.0072267\,f_1 f_2 - 0.0033663\,f_2^2$$
$$-0.0487802\,f_1^3 - 0.0343368\,f_1^2 f_2 + 0.0428779\,f_1 f_2^2 + \overset{+0.0100864}{0.0098974}\,f_2^3$$
$$+0.2203828\,f_1^4 + 0.8137909\,f_1^3 f_2 + \overset{+0.8353022}{0.8340265}\,f_1^2 f_2^2 + \overset{+0.4853356}{0.4841780}\,f_1 f_2^3$$
$$+\underset{+0.0868262}{0.0861086}\,f_2^4\Big\}$$

410

$$F\left[0.1535714\,F + 0.0570588\,G + \overset{0.09907239}{0.101092626}\,H + 0.010438597\,I - 0.101242236\,K\right]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
				+0.1535714
+0.0356618	+0.0481434	−0.1426470	−0.5777204	−0.2888602
			−0.0619702 −0.0631829	−0.0835923 −0.0852969
		+0.2348684	+0.4932237	+0.1438569
		−0.0395477	−0.1067789	−0.0720758

$$+0.0356618\,f_1 f_2^2 + 0.0481434\,f_2^3 + 0.0526737\,f_1^2 f_2^2 - 0.2544585\,f_1 f_2^3 - 0.1488046\,f_2^4$$

−0.2531958 −0.14710000

$$G\left[0.00531944\,G + 0.01824903\,H + 0.00195833\,I - 0.01916667\,K\right]$$ 412

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
+0.0033247	+0.0044883	−0.0132986	−0.0538593	−0.0269297
			−0.0114056	−0.0153976
		+0.0440625	+0.09253125	+0.0269828
		−0.0074870	−0.0202148	−0.0136450

$$+0.0033247\,f_1 + 0.0044883\,f_2 + 0.0232769\,f_1^2 + 0.0070516\,f_1 f_2 - 0.0289840\,f_2^2$$

$$+0.625\,f_1 + 0.84375\,f_2 - 2.5\,f_1^2 - 10.125\,f_1 f_2 - 5.0625\,f_2^2$$

$$\eta^2\Big\{0.0030779\,f_1^2 + 0.0056104\,f_1 f_2 + 0.0037870\,f_2^2 +$$
$$+ 0.0062363\,f_1^3 - 0.0208362\,f_1^2 f_2 - 0.0744405\,f_1 f_2^2 - 0.0471773\,f_2^3 -$$
$$- 0.0581923\,f_1^4 - 0.2533076\,f_1^3 f_2 - 0.1167768\,f_1^2 f_2^2 + 0.2577643\,f_1 f_2^3$$
$$+ 0.1467315\,f_2^4\Big\}$$

$H\left[0.0163730\, H + 0.00328644\, I - 0.03102659\, K \right]$

f_1^2	$f_1 f_2$	f_2^2
	-0.0102331	-0.0138147
$+0.0739449$	$+0.1552843$	$+0.0452913$
-0.0121198	-0.0327234	-0.0220883
$+0.0618251\, f_1^2$	$+0.1123278\, f_1 f_2$	$+0.0093883\, f_2^2$
× 0	$-0.625\, f_1 f_2$	$-0.84375\, f_2^2$

$$\eta^2 \left\{ -0.0386407\, f_1^3 f_2 - 0.1223698\, f_1^2 f_2^2 - 0.1006443\, f_1 f_2^3 - 0.0079214\, f_2^4 \right\}$$

$K\left[0.01845238\, K - 0.00362179\, I \right]$

f_1^2	$f_1 f_2$	f_2^2
$+0.0072080$	$+0.0194615$	$+0.0131365$
-0.0814903	-0.1711296	-0.0499128
$-0.0742823\, f_1^2$	$-0.1516681\, f_1 f_2$	$-0.0367763\, f_2^2$

×) $0.390625\, f_1^2 + 1.0546875\, f_1 f_2 + 0.7119140625\, f_2^2$

$$\eta^2 \left\{ -0.0290165\, f_1^4 - 0.1375900\, f_1^3 f_2 - 0.2272108\, f_1^2 f_2^2 - 0.1467622\, f_1 f_2^3 - 0.0261816\, f_2^4 \right\}$$

$$0.0001\, I^2 = \eta^2 \left\{ 0.225\, f_1^2 + 0.4725\, f_1 f_2 + 0.1378125\, f_2^2 \right\}^2$$

$$= \eta^2 \left\{ 0.050625\, f_1^4 + 0.212625\, f_1^3 f_2 + 0.28527187 5\, f_1^2 f_2^2 + 0.1302328\, f_1 f_2^3 + 0.0189923\, f_2^4 \right\}$$

$$\eta^2\, 0.0001821338\, I^2 = \eta^2 \left\{ 0.0922052\, f_1^4 + 0.3872620\, f_1^3 f_2 + 0.5195765\, f_1^2 f_2^2 + 0.2371979\, f_1 f_2^3 + 0.0345914\, f_2^4 \right\}$$

Terms with coefficient η

f_1^2	$f_1 f_2$	f_2^2
+0.1125000	+0.0864583	+0.0196875
−0.0960938	−0.0692637	−0.0148127

$$\eta \left(+0.0164062\, f_1^2 + 0.0171946\, f_1 f_2 + 0.0048748\, f_2^2 \right) \quad \text{from 61}$$
$$\;\; \left(+0.0304688\, f_1^2 + 0.0319329\, f_1 f_2 + 0.0090532\, f_2^2 \right) \quad \text{from } -80$$

$$\eta \left(+0.0468750\, f_1^2 + 0.0491275\, f_1 f_2 + 0.0139280\, f_2^2 \right)$$

η^2

f_1^2	$f_1 f_2$	f_2^2	f_1^3	$f_1^2 f_2$	$f_1 f_2^2$	f_2^3	
+0.00036874 -0.0053442	+0.0037879 -0.0062538	+0.0010068 -0.0014977					
+0.0030762 ✓	+0.0025635 ✓	+0.0005341 ✓	-0.0105469 ✓	-0.0125000 ✓	-0.0052230 ✓	-0.0007690 ✓	
-0.0061523 ✓	-0.0051270 ✓	-0.0010681 ✓	+0.01[illegible]557 ✓	+0.0227471 +0.0163186	+0.0093173 ✓	+0.0013477 ✓	
-0.0210938 ✓	+0.0098413 ✓	0 ✓	+0.0956318 +0.0955868	-0.0447635 -0.0448580	-0.0057587 -0.0057869	0 ✓	
0 ✓	-0.0603366 ✓	-0.0214089 ✓	0 ✓	+0.3705701 ✓	+0.2385724 ✓	-0.0049055	
+0.0279352 ✓	+0.0555468 ✓	+0.0287591 ✓	-0.0640917 ✓	-0.3648761 -0.0995788	-0.2678263 ✓	+0.0176204 ✓	
0 ✓	0 ✓	0 ✓	0 ✓	+0.0811607 ✓	+0.0254238 ✓	-0.0265850 ✓	
0 ✓	-0.0072267 ✓	-0.0033663 ✓	-0.0414802 ✓	-0.0343368 ✓	+0.0428779 ✓	+0.0100864 +0.0098974	
0 ✓	0 ✓	0 ✓	0 ✓	0 ✓	+0.0356618 ✓	+0.0481434 ✓	
+0.0020779 ✓	+0.0056104 ✓	+0.0037870 ✓	+0.0012363 ✓	-0.0708362 ✓	-0.0744405 ✓	-0.0471773 ✓	
+0.00121194 $-0.0045010 f_1^2$	+0.0046676 $-0.0053741 f_1 f_2$	+0.0033758 $-0.0012608 f_2^2$	-0.0020000 $-0.0020450 f_1^3$	-0.0028347 $+0.2559396 f_1^2 f_2$	-0.0013953 $-0.0014235 f_1 f_2^2$	$-0.0002389 f_2^3$	ε_1

$+0.0000529 f_1^2 + 0.0000612 f_1 f_2 + 0.0000177 f_2^2$ from ε_3

$+0.0012648 f_1^2 + 0.0047288 f_1 f_2 + 0.0033935 f_2^2$

415

η^2

f_1^4	$f_1^3 f_2$	$f_1^2 f_2^2$	$f_1 f_2^3$	f_2^4
+0.0090402	+0.0138951	+0.0085034	+0.0024316	+0.0002769
−0.0154436	−0.0230004	−0.0131381	−0.0037776	−0.0004166
−0.1068884 −0.1067984	+0.0509298 +0.0503188	+0.0115174 +0.0115738	0	0
0	−0.5168948	−0.4999912	+0.0176220	0
−0.1105758	+0.5707562	+0.8499994	−0.0985624	−0.1521878
0	−0.7854982	−1.0262181	−0.0981354	+0.1722641
+0.2203828	+0.8137909	+0.8353022 +0.8340265	+0.4853356 +0.4841780	+0.086262 +0.0861086
0	0	+0.0526737	−0.2531458 −0.2544585	−0.1471000 −0.1488046
−0.0581923	−0.2533076	−0.1167768	+0.2571643	+0.1467315
0	−0.0386407	−0.1223618	−0.1001443	−0.0079214
−0.0290165	−0.1375900	−0.2272108	−0.1467622	−0.0261816
+0.0922052	+0.3872620	+0.5195765	+0.2371979	+0.0345914

$+0.0015116$ / ~~$+0.0016011$~~ f_1^4 $+0.0809023$ / $+0.0810913$ $f_1^3 f_2$ $+0.2715418$ / $+0.2703225$ $f_1^2 f_2^2$ $+0.3012737\, f_1 f_2^3$ $+0.1063185$ / $+0.1044605\, f_2^4$ from $\mathcal{E}_1$

$+0.0003571\, f_1^4 + 0.0007484\, f_1^3 f_2 + 0.0006044\, f_1^2 f_2^2 + 0.0002224\, f_1 f_2^3 + 0.0000315\, f_2^4$ from $\mathcal{E}_3$

$$\left(+0.0018687 f_1^4 + 0.0816507 f_1^3 f_2 + 0.2721462 f_1^2 f_2^2 + 0.3014961 f_1 f_2^3 + 0.1063500 f_2^4\right)\eta^2$$

416

$$\frac{H}{R^3} = \frac{1}{2}\frac{\sigma^3}{E^2}\pi\left(\frac{t}{R}\right)\Big[\eta\,(0.4709575 f_1^2 + 0.5756148 f_1 f_2 + 0.1942698 f_2^2)\frac{1}{K^2}$$

$$- \eta^2 (0.0468750 f_1^2 + 0.0491275 f_1 f_2 + 0.0139280 f_2^2)$$

$$+ \eta^3 (0.0012648 f_1^2 + 0.0047288 f_1 f_2 + 0.0033935 f_2^2 - 0.0020000 f_1^3 - 0.0028347 f_1^2 f_2$$

$$- 0.0013953 f_1 f_2^2 - 0.0002389 f_2^3 + 0.0018687 f_1^4 + 0.0816507 f_1^3 f_2 + 0.2421462 f_1^2 f_2^2$$

$$+ 0.3014961 f_1 f_2^3 + 0.1063500 f_2^4)\Big]$$

If $f_2 = 0$, the conditions for equilibrium are

$$\frac{0.4709575}{K^2} - 0.0937500\,\eta + \eta^2 (0.0037944 - 0.0060000 f_1 + 0.0056061 f_1^2) = 0$$

$$\frac{0.9419150}{K^2} - 0.0937500\,\eta + \eta^2 (0.0025296 - 0.0060000 f_1 + 0.0074748 f_1^2) = 0$$

This set of equations can be put into the form

$$\eta^2 + A\eta + \frac{B}{K^2} = 0$$

$$\eta^2 + C\eta + \frac{D}{K^2} = 0$$

The resultant will be

$$\begin{vmatrix} C & \frac{D}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} - \begin{vmatrix} A & \frac{B}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} = 0$$

If $f_2 = 0$, the condition for equilibrium is

$$\frac{0.9419150}{K^2} - 0.0937500\eta + \eta^2(0.0025296 - 0.0060000 f_1 + 0.0074748 f_1^2) = 0.$$

where $\eta = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2$, Now $\frac{\delta}{R} = f_1 \cdot \frac{1}{2}\left(\frac{a}{R}\right)^2$

$$\frac{\delta}{t} = \mu = \frac{1}{2}\left(\frac{a}{R}\right)^2 \frac{f_1}{\left(\frac{t}{R}\right)}$$

or $$f_1 = \frac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}$$

Thus $$\frac{0.9419150}{K^2} - 0.0937500\left(\frac{E}{\sigma}\right)\left(\frac{a}{R}\right)^2 + \left(\frac{E}{\sigma}\right)^2\left(\frac{a}{R}\right)^4\left(0.0025296 - 0.006000\frac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}\right.$$
$$\left. + 0.0074748\frac{\mu^2\left(\frac{t}{R}\right)^2}{\frac{1}{4}\left(\frac{a}{R}\right)^4}\right) = 0$$

$$\frac{0.9419150}{K^2} - \frac{0.0937500}{K}\cdot\gamma^2 + \frac{1}{K^2}\left(0.0025296\gamma^4 - 0.012000\mu\gamma^2\right.$$
$$\left. + 0.0298992\mu^2\right) = 0.$$

Thus
$$0.09375K\gamma^2 = 0.0025296\gamma^4 - 0.012000\mu\gamma^2 + (0.941915 + 0.0298992\mu^2)$$

$$K = 0.026982\gamma^2 - 0.12800\mu + \frac{10.047 + 0.31892\mu^2}{\gamma^2}$$

$$\gamma^2 = \sqrt{\frac{10.047 + 0.31892\mu^2}{0.026982}}$$

$$K = 2\sqrt{0.026982\,(10.047 + 0.31892\,\mu^2)} - 0.12800\mu$$

$$= 0.3286\sqrt{10.047 + 0.31892\,\mu^2} - 0.12800\mu$$

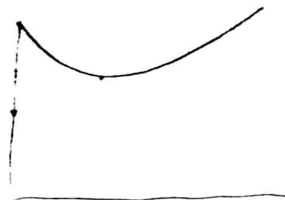
~~$\mu = 0.5$, $= 1.046$~~ $K = 0.1856\sqrt{31.503 + \mu^2} - 0.12800\mu$

$\mu = 2$, $K = 0.850$

$\mu = 4$, $K = 0.768$

$\mu = 6$, $K = 0.758$

$\mu = 7$, $K = 0.767$



$$\frac{M}{R^3}=\frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{t}{R}\right)\left(\frac{a}{R}\right)^2\Big\{ f_1^2\left(0.4709575+0.5756148\rho+0.1942698\rho^2\right)\frac{1}{K^2}$$

$$-\eta f_1^2\left(0.0468750+0.0491275\rho+0.0139280\rho^2\right)+\eta^2 f_1^2\Big[0.0012648+0.0042288\rho+0.0033935\rho^2$$

$$-f_1\left(0.0020000+0.0028347\rho+0.0013953\rho^2+0.0002389\rho^3\right)+f_1^2\left(0.0018687+0.0311507\rho+0.2721462\rho^2\right.$$

$$\left.+0.3014961\rho^3+0.1063500\rho^4\right)\Big]\Big\}$$

$$\rho=\frac{f_2}{f_1}$$

$$\left(0.9419150+1.1512296\rho+0.3885396\rho^2\right)\frac{1}{K^2}-\eta\left(0.0937500+0.0982550\rho+0.0278560\rho^2\right)$$

$$+\eta^2\Big[0.0025296+0.0094576\rho+0.0067870\rho^2-f_1\left(0.0060000+0.0085041\rho+0.0041859\rho^2+0.0007167\rho^3\right)$$

$$+f_1^2\left(0.0074748+0.3266098\rho+1.0885848\rho^2+1.2059844\rho^3+0.425400\rho^4\right)\Big]=0.$$

$$\left(0.5756148+0.3885396\rho\right)\frac{1}{K^2}-\eta\left(0.0491275+0.0278560\rho\right)$$

$$+\eta^2\Big[0.0042288+0.0067870\rho-f_1\left(0.0028347+0.0027906\rho+0.0007167\rho^2\right)$$

$$+f_1^2\left(0.0311507+0.5442924\rho+0.9044883\rho^2+0.425400\rho^3\right)\Big]=0.$$

420

$$\frac{1}{K^2}(0.9419150 + 0.5756148\,\rho) - \frac{1}{K}\gamma^2(0.093750 + 0.0491275\,\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0025296 + 0.0042288\,\rho - \frac{\mu_1}{\gamma^2}(0.0120000 + 0.0113388\,\rho + 0.0027906\,\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.0298992 + 0.9478084\,\rho + 2.1771696\,\rho^2 + 1.2059844\,\rho^3)\Big] = 0$$

$$\mu_1 = \frac{1}{2}\frac{\left(\frac{\rho}{R}\right)^2}{\left(\frac{t}{R}\right)} f_1 = \frac{\delta_1}{t}$$

$$\frac{1}{K^2}(0.5756148 + 0.3885396\,\rho) - \frac{1}{K}\gamma^2(0.0491275 + 0.027856\,\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0042288 + 0.0067870\,\rho - \frac{\mu_1}{\gamma^2}(0.0056694 + 0.0055812\,\rho + 0.0014334\,\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.3266028 + 2.1771196\,\rho + 3.6179532\,\rho^2 + 1.701600\,\rho^3) = 0.$$

$$(0.0298992 + 0.9478084\,\rho + 2.1771696\,\rho^2 + 1.2059844\,\rho^3)\mu_1^2 - \gamma^2(0.012 + 0.0113388\,\rho + 0.0027906\,\rho^2)\mu_1$$

$$+ \Big[0.941915 + 0.5756148\,\rho + (0.0025296 + 0.0042288\,\rho)\gamma^4 - K\gamma^2(0.093750 + 0.0491275\,\rho)\Big] = 0.$$

$$(0.3266028 + 2.1771196\,\rho + 3.6179532\,\rho^2 + 1.701600\,\rho^3)\mu_1^2 - \gamma^2(0.0056694 + 0.0055812\,\rho + 0.0014334\,\rho^2)\mu_1$$

$$+ \Big[0.5756148 + 0.3885396\,\rho + (0.0042288 + 0.0067870\,\rho)\gamma^4 - K\gamma^2(0.0491275 + 0.027856\,\rho)\Big] = 0$$

421

$$\mu = \left(\frac{\ell}{t}\right) = 0.375\,\mu_1 + 0.15625\,\mu_0 = \mu_1\,(0.375 + 0.15625\,\rho)$$

$$\frac{\frac{\mu}{\mu_1} - 0.375}{0.15625} = \rho = 6.4000\,\frac{\mu}{\mu_1} - 2.08000$$

$$\mu_1^3\Big[0.0298992 + 0.9798084\left(6.4000\,\frac{\mu}{\mu_1} - 2.08000\right) + 2.1771696\left(40.96\,\frac{\mu^2}{\mu_1^2} - 26.624\,\frac{\mu}{\mu_1} + 4.3264\right)$$
$$+ 1.2059844\left(262.144\,\frac{\mu^3}{\mu_1^3} - 255.5904\,\frac{\mu^2}{\mu_1^2} + 83.06688\,\frac{\mu}{\mu_1} - 8.998912\right)\Big]$$

$$= \left[-3.4413432\,\mu_1^3 + 48.483172\,\mu_1^2\mu - 219.06117\,\mu_1\mu^2 + 316.14157\,\mu^3\right]$$

$$\mu_1^2\Big[0.012 + 0.0113388\left(6.4000\,\frac{\mu}{\mu_1} - 2.08000\right) + 0.0022[illegible]6\left(40.96\,\frac{\mu^2}{\mu_1^2} - 26.624\,\frac{\mu}{\mu_1} + 4.3264\right)\Big]$$

$$= \left[0.0004885\,\mu_1^2 - 0.0017286\,\mu_1\mu + 0.1143030\,\mu^2\right]$$

$$\mu_1\left[0.941915 + 0.578148\left(6.400\,\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.2553638\,\mu_1 + 3.6839347\,\mu$$

$$\mu_1\left[0.0025296 + 0.0047288\left(6.400\,\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0073063\,\mu_1 + 0.0302643\,\mu$$

$$\mu_1\left[0.93750 + 0.0491275\left(6.400\,\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0084352\,\mu_1 + 0.3144160\,\mu\,.$$

$$\mu_1^3\left[0.3266028 + 2.1771696\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 3.6179532\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)\right.$$

$$\underline{\underline{423}}$$

$$\left. + 1.701600\left(262.144\frac{\mu^3}{\mu_1^3} - 255.5904\frac{\mu^2}{\mu_1^2} + 83.06688\frac{\mu}{\mu_1} - 8.998912\right)\right]$$

$$= -3.8617459\,\mu_1^3 + 58.9561025\,\mu_1^2\mu - 286.72126\,\mu_1\mu^2 + 446.06423\,\mu^3$$

$$\mu_1^2\left[0.0056694 + 0.0055812\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 0.0014334\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)\right]$$

$$= \left[0.0002620\,\mu_1^2 - 0.0024432\,\mu_1\mu + 0.0587121\,\mu^2\right]$$

$$\mu_1\left[0.5756148 + 0.3885396\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.2325476\,\mu_1 + 2.4866534\,\mu$$

$$\mu_1\left[0.0047288 + 0.0067870\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0093882\,\mu_1 + 0.0434368\,\mu$$

$$\mu_1\left[0.0491275 + 0.027856\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0088130\,\mu_1 + 0.1782784\,\mu$$

$$A_1\gamma^4 + B_1\gamma^2 + C_1 = 0$$

$$A_2\gamma^4 + B_2\gamma^2 + C_2 = 0$$

$$C_1 = +3.4413432\mu_1^3 - 48.4831722\mu_1^2\mu + 219.0611\mu_1\mu^2 - 316.14157\mu^3 + 0.2553638\mu_1 - 3.6839347\mu$$

$$B_1 = \{0.0004885\mu_1^2 - 0.0017286\mu_1\mu + 0.1143030\mu^2 - K(0.0084353\mu_1 - 0.3144160\mu)\}$$

$$A_1 = (0.0043063\mu_1 - 0.0302643\mu)$$

$$C_2 = +3.8617459\mu_1^3 - 58.9561025\mu_1^2\mu + 286.721\mu_1\mu^2 - 446.06423\mu^3 + 0.2325426\mu_1 - 2.4866534\mu$$

$$B_2 = \{0.0002620\mu_1^2 - 0.0024432\mu_1\mu + 0.0587121\mu^2 - K(0.0088130\mu_1 - 0.1782784\mu)\}$$

$$A_2 = (0.0093882\mu_1 - 0.0434368\mu)$$

The resultant for elimination of f^2 is

$$(A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1) = 0$$

$\mu = 0$

$A_1 = 0.0073063\,\mu_1$

$B_1 = 0.0004885\,\mu_1^2 - 0.0084352\,\mu_1 K$

$C_1 = 3.4413432\,\mu_1^3 + 0.2553638\,\mu_1$

$A_2 = 0.0093882\,\mu_1$

$B_2 = 0.0002620\,\mu_1^2 - 0.0088130\,\mu_1 K$

$C_2 = 3.8617459\,\mu_1^3 + 0.2325476\,\mu_1$

$$(A_1 C_2 - A_2 C_1)^2 = \mu_1^4 (-0.0040929\,\mu_1 - 0.00069834)^2$$

$$(A_1 B_2 - A_2 B_1) = \mu_1^2 (-0.0000026719\,\mu_1 - 0.0000148009\,K)$$

$$(B_1 C_2 - B_2 C_1) = \mu_1^2 [\,0.0009848\,\mu_1^3 + 0.00004669\,\mu_1 - (0.0022460\,\mu_1^2 - 0.00028894)K\,]$$

$$(40.929\,\mu_1 + 6.9834)^2 + (0.26719\,\mu_1 + 1.48009\,K) \times$$
$$\times [\,0.9848\,\mu_1^3 + 0.04669\,\mu_1 - (2.2460\,\mu_1^2 - 0.28894\,K\,] = 0$$

$$1675.183\,\mu_1^2 + 571.647\,\mu_1 + 48.7679$$

Let $\xi = \frac{\mu_1}{\mu}$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = \mu\left(0.0004885\,\xi^2 - 0.0017286\,\xi + 0.1143030\right) - K\left(0.0084352\,\xi - 0.3144160\right)$$

$$C_1 = \mu^2\left(3.4413432\,\xi^3 - 48.4831792\,\xi^2 + 219.06117\,\xi - 316.14157\right) + \left(0.2553638\,\xi - 3.6839147\right)$$

$$A_2 = 0.0093882\,\xi - 0.0434368$$

$$B_2 = \mu\left(0.0002620\,\xi^2 - 0.0024432\,\xi + 0.0587121\right) - K\left(0.0088130\,\xi - 0.1782784\right)$$

$$C_2 = \mu^2\left(3.8617459\,\xi^3 - 58.9561025\,\xi^2 + 286.72126\,\xi - 446.06423\right) + \left(0.2325476\,\xi - 2.4866534\right)$$

Let $\mu = 7$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = 0.0034195\,\xi^2 - 0.0121002\,\xi + 0.800121 - K\left(0.0084352\,\xi - 0.3144160\right)$$

$$C_1 = 168.62582\,\xi^3 - 2375.6754\,\xi^2 + 10733.99733\,\xi - 15490.93693 + 0.2553638\,\xi - 3.6839147$$

$$C_1 = 168.62582\,\xi^3 - 2375.6754\,\xi^2 + 10734.2529\,\xi - 15494.621$$

$$A_2 = 0.0093882\,\xi - 0.0434368$$

$$B_2 = 0.001834\,\xi^2 - 0.0171024\,\xi + 0.4109847 - K\left(0.0088130\,\xi - 0.1782784\right)$$

$$C_2 = 189.22555\,\xi^3 - 2888.8490\,\xi^2 + 14049.575\,\xi - 21859.634$$

We put

$A_1 = a_1 \qquad A_2 = a_2$

$B_1 = b_1 - c_1 K \qquad B_2 = b_2 - c_2 K$

$C_1 = d_1 \qquad C_2 = d_2$

$$\therefore \quad (a_1 d_2 - a_2 d_1)^2 - [a_1 b_2 - a_2 b_1 - K(a_1 c_2 - a_2 c_1)][b_1 d_2 - b_2 d_1 - K(c_1 d_2 - c_2 d_1)] = 0$$

$$\therefore \{(a_1 d_2 - a_2 d_1)^2 - (a_1 b_2 - a_2 b_1)(b_1 d_2 - b_2 d_1)\}$$

$$+ \{(a_1 c_2 - a_2 c_1)(b_1 d_2 - b_2 d_1) + (a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)\} K$$

$$+ (a_1 c_2 - a_2 c_1)(c_1 d_2 - c_2 d_1) K^2 = 0$$

	$\xi = -10$		0		$\xi = +10$			
	1	2	1	2	1	2		
a	−0.1033273	−0.1373188	−0.0302643	−0.0434368	+0.0427987	+0.0504452		
b	+1.263073	+0.7654087	+0.800121	+0.4109847	+1.0210690	+0.4233607		
c	−0.398768	−0.2664084	−0.3144160	−0.1782784	−0.2300640	−0.0901484		
d	−52903.050	−64046.83	−15494.621	−21859.634	+22906.18	+18976.77		
$a_1d_2 - a_2d_1$	−6468.228	(1)	−11.4702	(1)	−343.326	(1)		
$a_1b_2 - a_2b_1$	+0.0943561	(2)	+0.0223165	(2)	−0.0333888	(2)		
$b_1d_2 - b_2d_1$	−404030.71	(3)	−11122.300	(3)	+9679.015	(3)		
$a_1c_2 - a_2c_1$	−0.0272311	(4)	−0.0082617	(4)	+0.0077474	(4)		
$c_1d_2 - c_2d_1$	+114459.11	(5)	+4110.6624	(5)	−2300.916	(5)		
$(1)^2 - (2)(3)$	41876096	(6)	379.776	(6)	118175.9	(6)		
$(3)(4) + (2)(5)$	21802.116	(7)	183.625	(7)	151.812	(7)		
$(4)(5)$	−3116.847	(8)	−33.9613	(8)	−17.8261	(8)		
$(7)/(8)$	−6.99493	(9)	−5.40689	(9)	−8.51628	(9)		
$(6)/(8)$	−13435.403	(10)	−11.18261	(10)	−6630.497	(10)		
$-(9)/2$			+2.70345	(11)	+4.25814	(11)		
$(11)^2 - (10)$			+18.49125	(12)	+6648.63	(12)		
$(12)^{1/2}$			+4.30015	(13)	+81.5391	(13)		
$(11) + (13)$			+7.00360	K	+85.79726	K		

	function f_1	function f_2	function f_3	$0.625 f_3$	$f_1-0.625f_3$	W_1	$0.84375 f_3$	$f_2-0.84375f_3$	W_2		
0	1.0000	1.00000	1.00000	0.6250000	0.3750000	1.000000	0.84375000	0.15625000	1.000000		
0.1	0.98010	0.998001	0.99980001	0.62487501	0.35522499	0.947267	0.84358126	0.15441974	0.988286		
0.2	0.92160	0.984064	0.99680256	0.62300160	+0.29859840	0.796262	0.84105216	0.14301184	0.915276		
0.3	0.82810	0.946729	0.98386561	0.61491601	+0.21318399	0.568491	0.83013661	0.11659239	0.746191		
0.4	0.70560	0.876096	0.94945536	0.59340960	+0.11219040	0.299174	0.80110296	0.07499304	0.479955		
0.5	0.56250	0.765625	0.87890625	0.54931641	+0.01318359	0.035156	0.74157715	0.02404785	0.153906		
0.6	0.40960	0.614656	0.75759616	0.47349760	-0.06389760	-0.170394	0.63922176	-0.0245686	-0.157221		
0.7	0.26010	0.431649	0.57744801	0.36090501	-0.10080501	-0.268813	0.48722176	-0.05557276	-0.355666		
0.8	0.12960	0.238144	0.34857216	0.21785760	-0.08825760	-0.235354	0.29410776	-0.05576376	-0.358168		
0.9	0.03610	0.073441	0.11826721	0.07391701	-0.03781701	-0.100845	0.09978796	-0.02634696	-0.168621		

The difficulty now is evidently that the wave form W_1, W_2 will not give a single-wave buckle.

With $\nu = 0$, the equations on p. 383 become

$$q_2 + 0.666667\, s_2 - 0.083333 = \eta\left\{0.333333\, p_2 + 0 + 0.00173611\, f_1 - 0.000[illegible]\, f_2 - 0.00138889\, f_3\right\}$$

$$q_2 + 0 - 0.0833333 = \eta\left\{0.333333\, p_2 + 2r_2 - 0.02604167\, f_1 + 0.01845238\, f_2 + 0.02083333\, f_3\right\}$$

$$q_2 + 0.333333\, s_2 + 0.0833333 = \eta\left\{-0.33333\, p_2 - r_2 + 0.00868056\, f_1 - 0.01145575\, f_2 - 0.0111111\, f_3\right\}$$

$$q_2 + 2\, s_2 + 0.250000 = \eta\left\{-p_2 + 0 - 0.06770833\, f_1 - 0.07946429\, f_2 - 0.08750000\, f_3\right\}$$

$$q_2 + 0 - 0.250000 = \eta\left\{+p_2 + 3r_2 - 0.005208333\, f_1 + 0.06741071\, f_2 + 0.0750000\, f_3\right\}$$

$$0.66666667\, s_2 + 0 = \eta\left\{0 - 9r_2 + 0.02777778\, f_1 - 0.01875000\, f_2 - 0.[illegible]222222\, f_3\right\}$$

$$0.3333333\, s_2 + 0.16666667 = \eta\left\{-0.6666667\, p_2 - 3r_2 + 0.03472222\, f_1 - 0.03020833\, f_2 - 0.03194444\, f_3\right\}$$

$$1.66666667\, s_2 + 0.16666667 = \eta\left\{-0.666667\, p_2 + r_2 - 0.07638889\, f_1 - 0.06770834\, f_2 - 0.076388889\, f_3\right\}$$

$$2\, s_2 + 0.500000 = \eta\left\{-2p_2 - 3r_2 - 0.06250000\, f_1 - 0.14687500\, f_2 - 0.16250000\, f_3\right\}$$

$$s_2 + 0 = \eta\left\{0 - 3r_2 + 0.04166667\, f_1 - 0.02812500\, f_2 - 0.0333333\, f_3\right\}$$

$$s_2 + 0.50000 = \eta\left\{-9p_2 - 7\,[illegible]\, f_1 - 0.09062500\, f_2 - 0.09583333\, f_3\right\}$$

$$s_2 + 0.10000 = \eta\left\{-0.4p_2 + 0.6r_2 - 0.04583333\, f_1 - 0.04062500\, f_2 - 0.04583333\, f_3\right\}$$

$$s_2 + 0.25000 = \eta\left\{-p_2 - 1.5r_2 - 0.03125000\, f_1 - 0.07343750\, f_2 - 0.08125000\, f_3\right\}$$

$$0.5000000 = \eta\{-2p_2 - 6r_2 + 0.06250000 f_1 - 0.06250000 f_2 - 0.06250000 f_3\}$$

$$0.400000 = \eta\{-1.6p_2 - 9.6r_2 + 0.15000000 f_1 - 0.05000000 f_2 - 0.05000000 f_3\}$$

$$0.150000 = \eta\{-0.6p_2 - 2.1r_2 + 0.01458333 f_1 - 0.03281250 f_2 - 0.03541667 f_3\}$$

$$0.2500000 = \eta\{-p_2 - 3r_2 + 0.03125000 f_1 - 0.03125000 f_2 - 0.03125000 f_3\}$$

$$0.250000 = \eta\{-p_2 - 6r_2 + 0.09375000 f_1 - 0.03125000 f_2 - 0.03125000 f_3\}$$

$$0.250000 = \eta\{-p_2 - 3.5r_2 + 0.02430555 f_1 - 0.05468750 f_2 - 0.05902778 f_3\}$$

$$3r_2 = 0.06250000 f_1 + 0 + 0$$

$$2.5r_2 = 0.06944444 f_1 + 0.02343750 f_2 + 0.02777778 f_3$$

$$3r_2 = 0.06250000 f_1$$

$$3r_2 = 0.08333333 f_1 + 0.02812500 f_2 + 0.03333333 f_3$$

$$0 = 0.02083333 f_1 + 0.02812500 f_2 + 0.03333333 f_3$$

$$-f_3 = 0.625000 f_1 + 0.843750 f_2$$

Same thing !!!

431

［General　Information］
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